Classification
*Physics Abstracts*
61.50E — 02.10 — 02.40

# Representation of certain self-similar quasiperiodic tilings with perfect matching rules by discrete point sets

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*(Received 24 August 1993, revised 21 January 1994, accepted 8 March 1994)*

**Abstract.** — Simple quasiperiodic tilings with 8-fold and 12-fold symmetry are presented that possess local de-/inflation symmetry and perfect matching rules. The special feature of these tilings is that the full information is already derivable from the set of vertex sites alone. This means that the latter is a valid representative of the corresponding equivalence class of mutual local derivability.

## 1. Introduction.

The existence of perfect matching rules for several quasiperiodic tilings [1-5] is interesting both mathematically and physically. On the one hand, one obtains so-called aperiodic sets (like the two rhombi of the Penrose pattern [1, 6]), on the other hand they offer possible geometric approaches to certain types of long range orientational order, as it occurs in quasicrystals, compare [7]. While the former is a well established branch of discrete geometry, the latter needs both further explanation and exploration.

Several types of matching rules have been discussed [8], in particular strong matching rules (enforcing aperiodicity) and perfect matching rules (uniquely specifying a local isomorphism class). Perfect matching rules for nonperiodic tilings are automatically strong while the converse is not necessarily true. In this article we focus on perfect matching rules which have the advantage that they are applicable to the periodic case as well: periodic tilings possess perfect matching rules in a trivial way via the repetition rule of the fundamental cell. This can simultaneously be used as a local growth rule whereby one obtains larger and larger patches from a suitable seed.

The aperiodic case is more complicated though: perfect matching rules are not automatically local growth rules. Even worse, there is — to our knowledge — no example known where they are. Consequently, in comparison with the periodic case, possessing matching rules now is a weaker property from the physical point of view. Nevertheless, one important property remains: perfect matching rules ensure the energetic stabilization of quasiperiodic structures as the ground state of a suitable Hamiltonian, by favouring finitely many local configurations relative to others, compare the discussion in [9].

In view of the local nature of physical interactions, this statement requires some care: If matching rules are significant, they must follow locally from the physical structure, ideally from the set of atomic positions, say. In geometric terms, the simplest such sets of points are the vertex sets of tilings. The question then is whether perfect matching rules can be derived locally from the set of vertex sites. This is the case for the Penrose and the decagonal triangle tiling [1, 10], but not for the Ammann-Beenker, the Socolar, and the Gähler-Niizeki tiling [2, 11, 3, 12, 4]. (We restrict the discussion to planar tilings here; for some general results we refer to [5, 13] and the remarkable article by Lunnon and Pleasants [14]).

Until recently, several researchers believed that perfect matching rules cannot be obtained for 8- or 12-fold symmetric tilings without introducing decorations which are not locally derivable from the bare tilings. This has been disproved by two counterexamples [10]. For the tilings $\mathcal{T}^{(8)}_{\mathrm{D}_4}$ and $\mathcal{T}^{(12)}_{\mathrm{D}_4}$ the existence of local de-/inflation symmetry and perfect matching rules was shown [15, 10], the latter by the composition/decomposition method [4]. One important result is that these properties automatically extend to all members of the equivalence class of mutual local derivability with symmetry preservation [16], called SMLD class from now on. For the concept and its properties we refer to [17]. To be more explicit: the existence of perfect matching rules is a property of an SMLD class, and can then be formulated for any of its members (being a class of local isomorphism – LI class). Now it is obvious that we can choose a point set representative out of the SMLD class, and the matching rules are then given as a finite list of possible patches up to a certain (finite) size. Though this might not seem most practical, it is certainly very close to the idea to stabilize a quasicrystal by the selection of finitely many clusters of atoms. Such an approach has been tried for a decagonal T-phase by a decoration of the decagonal triangle tiling [7]. There, the decoration enforces the matching rules, and it is therefore an explicit example of physical relevance, compare the discussion in [9].

To match physical applications, the selection of a point set representative of an SMLD class will prove useful — and it is *always* possible [9, 18]. Consequently, the formulation of matching rules, if existent, can be done in this framework, and quite a bit is known about the icosahedral [19] and the decagonal case [10, 7, 9]. In this article we are interested in 8- and 12-fold symmetry where still much less is known. For the 8- and 12-fold tilings in [15], the set of vertex sites alone does *not* represent the SMLD class. Therefore, it was an obvious exercise to find and describe other members of the SMLD class which *are* derivable from their vertex sites alone. A short presentation of two such tilings is the aim of the present article, though our generation procedure applies to a much broader class of cases. We thus continue [10], in particular the appendix of it.

## 2. The eightfold case: $\mathcal{P}^{(8)}$

Several different 8-fold tilings by squares and rhombi are known. Firstly, there is the standard octagonal square-rhombus tiling, found independently by Ammann [2] and Beenker [11] and also by Lück, in the latter case with various other examples obtained by substitution [20]. Recently, this tiling was put on a broader basis by investigation of the possible 2-tile substitutions with square and rhombus [21]. Thereby, two more tilings of the mentioned kind were found. Our tiling is still another one, because, in terms of the substitution formalism, different decompositions for both tiles occur, i.e. the generation by means of substitution is no longer one by two tiles only (although only two shapes of tiles occur).

But the way we have found it differs, wherefore we invite those who are interested in the

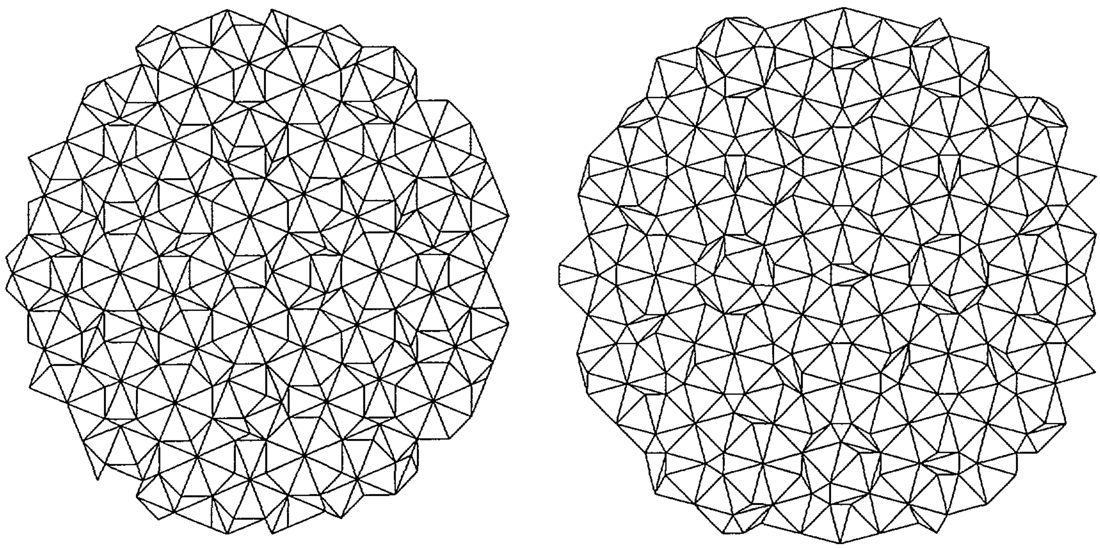

Fig. 1. — The quasiperiodic tilings $\mathcal{T}_{\mathrm{D}_4}^{(8)}$ and $\mathcal{T}_{\mathrm{D}_4}^{(12)}$

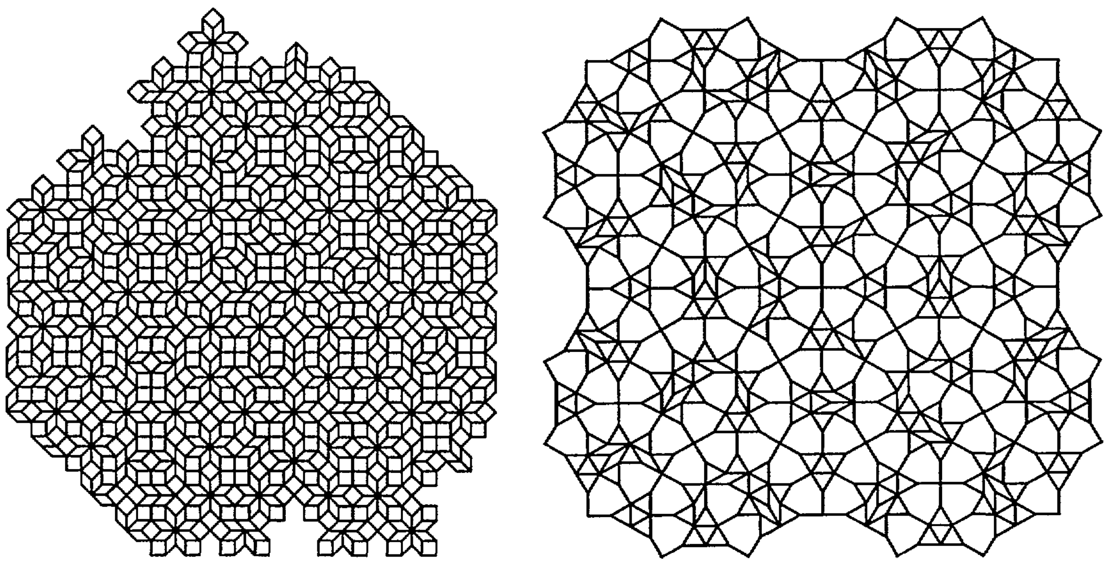

Fig. 2. — The quasiperiodic tilings $\mathcal{P}^{(8)}$ and $\mathcal{P}^{(12)}$. The patches shown are exact deflations of octagonally resp. dodecagonally shaped start configurations.

substitutional point of view to have a look into the Appendix. Let us start here with another 8-fold tiling, the tiling $\mathcal{T}_{\mathrm{D}_4}^{(8)}$ which is shown in figure 1. This tiling can be derived from the 4D root lattice $\mathrm{D}_4$ by the projection method [15]. It has a local deflation and inflation and possesses locally derivable matching rules [10]. On the other hand, the full local information

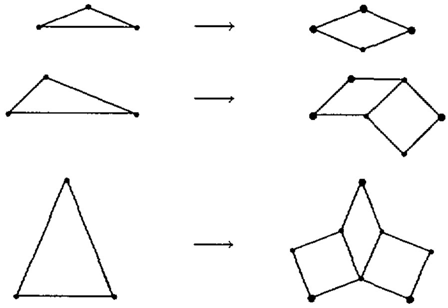

Fig. 3. — The local derivation of $\mathcal{P}^{(8)}$ from $\mathcal{T}_{D_4}^{(8)}$.

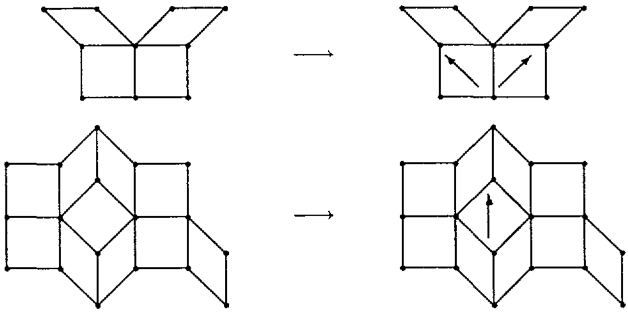

Fig. 4. — The decoration rule for $\mathcal{P}^{(8)}$ See text for additional explanation.

requires the knowledge of the *edges* in the tiling, the set of vertex sites alone is insufficent. In the present context, our task is to introduce additional points which complete the local information encoded in the points. Let us describe one possible procedure.

Firstly, one gets rid of the edge-inherent additional information by introducing new vertices. Especially hexagonal patches, compare figure 1, consisting of one acute triangle in the middle surrounded by two oblique ones at the cathedes and a flat triangle at the base, have to be decorated by new vertices in such a way that the decomposition by those edges is uniquely reflected within their sites. Therefore, we introduce quite near the base of the acute triangle an additional vertex. To get more handsome tiles we decorate the cathedes of the acute triangle with two additional vertices. The resulting derivation is shown in figure 3, its locality is obvious.

Next, we prove the local invertibility of this derivation. In order to do this, we introduce an orientation of the squares. Clearly, this orientation must be a local one: this is shown in figure 4. Just one ambiguous case remains: The patches of the lower row of figure 4 (without the lateral rhombus) may occur succeedingly up to three times in a line. In that case, it suffices to demand that the inner patch should be directed in the same way as the outer ones.

Now, having decorated the squares, the backward direction becomes easy: it is just the rule shown in figure 5. It is obvious that the flat triangles are regained by these rules as well.

The resulting 8-fold tiling $\mathcal{P}^{(8)}$ is again face-to-face and consists of squares and 45°-rhombi only. As an immediate consequence, this tiling is locally derivable from its vertex sites alone. Put together with the results of [10] one finds: the tiling $\mathcal{P}^{(8)}$ has perfect matching rules as well. Furthermore, the complete information is localy derivable from the set of vertex sites.

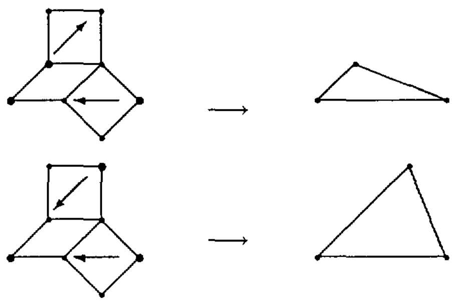

Fig. 5. — The remaining derivation from $\mathcal{P}^{(8)}$ back to $\mathcal{T}^{(8)}_{\mathrm{D}_4}$

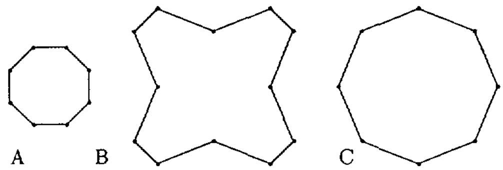


Fig. 6. — The acceptance domains for the tiling $\mathcal{P}^{(8)}$. The tetrasymmetric one occurs in two orientations. Domains A and B are situated around interstitials ('holes', more precise: A around those with integer Euclidean coordinates), whereas C is the one around lattice points. The projection of a unit vector $e_i$ would point from the lower of the left vertices of A, for example, towards the central point.

Especially:

The set of vertices of $\mathcal{P}^{(8)}$ has perfect matching rules.

For the sake of completeness, we will give the acceptance domains for the new tiling described above, where we use an embedding into the root lattice $\mathrm{D}_4$, the 4D checkerboard lattice [15]. Starting from the primitive hypercubic lattice with the standard orthonormal basis vectors $e_i$, we can write the lattice as

$$\mathrm{D}_4 = \{ \textstyle\sum x_i e_i \mid \sum x_i \equiv 0(2) \}.$$

In order to preserve the 8-fold symmetry within the 2-dimensional subspaces, the projector into the physical space is chosen as

$$\pi_{\|} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \sqrt{\frac{1}{2}} & \frac{1}{2} \end{pmatrix},$$

and for the projector into the internal space the second and fourth column have to be multiplied by $-1$.

Figure 6 shows the acceptance domains of the tiling $\mathcal{P}^{(8)}$ The small octagon (A) and the inner squares of the tetrasymmetric ones (B), defined by the concave vertices of them, correspond to the original tiling $\mathcal{T}^{(8)}_{\mathrm{D}_4}$, compare [15].

The corresponding vertex configurations are shown in Figure 7. The vertices A1 – A3 stem from the small octagonally shaped acceptance domain A, and are, therefore, deducible from

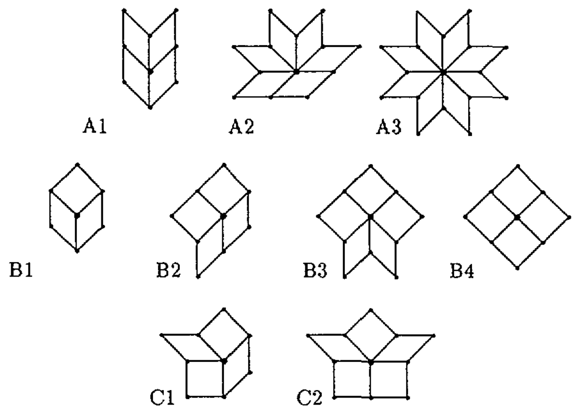


Fig. 7. — The 9 vertex configurations of the tiling $\mathcal{P}^{(8)}$.

Table I. — *Relative frequencies of occurrence for vertex configurations within the octagonal tiling* $\mathcal{P}^{(8)}$, $\lambda = 1 + \sqrt{2}$ *is the silver mean.*

| Vertex | Frequency of occurrence | | | |
|---|---|---|---|---|
| A1 | $1/2\lambda^4$ | $= (29 - 12\lambda)/2$ | $\approx$ | 1.47187 % |
| A2 | $(1+\lambda)/2\lambda^5$ | $= (-41 + 17\lambda)/2$ | $\approx$ | 2.08153 % |
| A3 | $(1+\lambda)/2\lambda^4$ | $= (17 - 7\lambda)/2$ | $\approx$ | 5.02526 % |
| B1 | $1/2$ | $= 1/2$ | $\approx$ | 50.00000 % |
| B2 | $1/\lambda^3$ | $= -12 + 5\lambda$ | $\approx$ | 7.10678 % |
| B3 | $1/2\lambda^4$ | $= (29 - 12\lambda)/2$ | $\approx$ | 1.47187 % |
| B4 | $1/2\lambda^3$ | $= (-12 + 5\lambda)/2$ | $\approx$ | 3.55339 % |
| C1 | $(1+\lambda)/2\lambda^4$ | $= (17 - 7\lambda)/2$ | $\approx$ | 5.02526 % |
| C2 | $(1+\lambda)/\lambda^3$ | $= -7 + 3\lambda$ | $\approx$ | 24.26407 % |

the corresponding vertex figures of $\mathcal{T}^{(8)}_{\mathrm{D}_4}$, B1 – B4 are mostly in one-to-one correspondence to those of $\mathcal{T}^{(8)}_{\mathrm{D}_4}$, just some vertices of type B1 occur additionally, namely those new vertices, in comparison to $\mathcal{T}^{(8)}_{\mathrm{D}_4}$, which are located at the edges of the acute triangles. Their sites do occur in the deflations of $\mathcal{T}^{(8)}_{\mathrm{D}_4}$ This is why the fourfold symmetric acceptance domains B do appear enlarged with respect to $\mathcal{T}^{(8)}_{\mathrm{D}_4}$ Worth noticing is the fact that the relative frequency of the vertices of type B1 within tiling $\mathcal{P}^{(8)}$ is exactly 50%, cf. table I. The last two vertex configurations are those which occur, with respect to $\mathcal{T}^{(8)}_{\mathrm{D}_4}$, additionally within the acute triangles. Their acceptance domain is the larger octagon C.

## 3. The twelvefold case: $\mathcal{P}^{(12)}$

Just as before, we transformed a $\mathrm{D}_4$-tiling by introducing new vertex sites. Again we started with a hexagonal patch, this time consisting of an acute triangle in the middle, two flat ones at its cathedes and a small equilateral triangle — all of them clearly from the twelvefold tiling

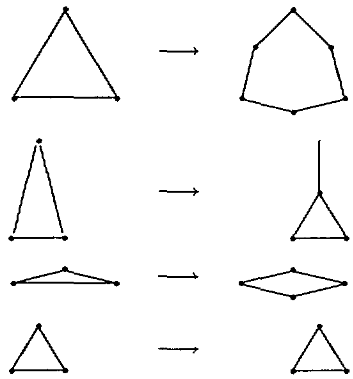

Fig. 8. — The local derivation of $\mathcal{P}^{(12)}$ from $\mathcal{T}_{D_4}^{(12)}$

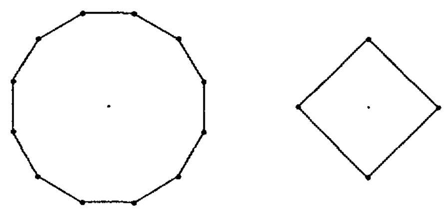

Fig. 9. — The acceptance domains of the tiling $\mathcal{P}^{(12)}$. The square occurs in three orientations. It is situated around interstitials ('holes'), whereas the other one occurs around lattice places. The projection of a unit lattice vector would point, for example, from a vertex of the square towards its center.

$\mathcal{T}_{D_4}^{(12)}$ [15], compare figure 1. Here, a single additional vertex site at the circumcenter of the acute triangles was enough to complete the local information. The vertex points of the tiling, $\mathcal{P}^{(12)}$, obtained this way turn out to be the union of the vertex sites of $\mathcal{T}_{D_4}^{(12)}$ and those of a tiling of Niizeki/Mitani and (independently) Gähler [12]. Therefore the vertex acceptance domains are those of these tilings, cf. figure 9. Up to this point, this was already shortly mentioned in the appendix of reference [10].

Now we have to prove mutual local derivability. The direction from $\mathcal{T}_{D_4}^{(12)}$ to $\mathcal{P}^{(12)}$ is an easy tile-to-tile derivation shown in figure 8. Due to the fact that the tiling $\mathcal{T}_{D_4}^{(12)}$ is already defined by the set of acute triangles alone, i.e. all the edges of the other tiles are given from those of the acute triangles, only these triangles need to by regained for the other direction. This looks difficult, because figure 8 indicates that the acute triangles as well as the small equilateral ones result in the same tiles. But again, due to the mentioned property of $\mathcal{T}_{D_4}^{(12)}$, it is clear that the small triangles of $\mathcal{T}_{D_4}^{(12)}$ have a unique surrounding by three acute triangles. This '3-star' patch will be transformed into a 'pyramid' of four (small) triangles. So it becomes clear that, whenever such a pyramid configuration occurs in $\mathcal{P}^{(12)}$, the central one does not stem from an acute triangle.

Like in the 8-fold case, we thus get a tiling with only one edge-length. The latter is the second shortest distance between vertex sites. Only the small diagonal of the rhombus is shorter, but

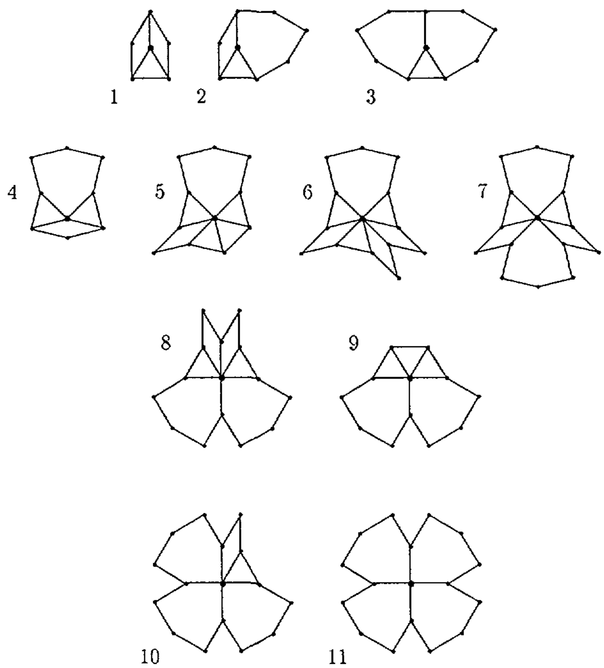


Fig. 10. — The vertex configurations of tiling $\mathcal{P}^{(12)}$. Vertices 1 through 3 correspond to the dodecagonal acceptance domain, whereas the others belong to the quadratic one(s).

no bond. On the other hand, this point distance occurs only between two vertex sites if these are connected by an edge, except for the single case that two rhombi are joined together. From the list of vertex configurations, shown in figure 10, it is obvious that more than two rhombi are not allowed to join. Therefore, we have to handle just this single exceptional patch. Here, the presence of the common rhombus vertex in between is enough to rule out that case. Thus, the complete information on the tiling $\mathcal{P}^{(12)}$ is contained in its vertices. Having shown the equivalence to the tiling $\mathcal{T}_{\mathrm{D}_4}^{(12)}$ as well and using again the results of [10], we find:

> The tiling $\mathcal{P}^{(12)}$ has perfect matching rules and its complete information is locally derivable from the set of vertex sites.

We present, in figure 9, the acceptance domains for $\mathcal{P}^{(12)}$. The quadratic domains correspond to $\mathcal{T}_{\mathrm{D}_4}^{(12)}$, the dodecagonal one to the tiling of references [12, 4]. In figure 10, the corresponding vertex configurations are shown. The first three configurations correspond to the new vertices, i.e., their acceptance domain is the dodecagonal one, the one of the (sub-)tiling shown in [12, 4], whereas the others correspond one-to-one to those of $\mathcal{T}_{\mathrm{D}_4}^{(12)}$ [15]. From the relative size of the dodecagonal acceptance domain with respect to the quadratic ones — the circumradius of the dodecagon equals the edge of the sqare — it becomes clear that the vertices $1-3$ have a total frequency of occurrence of 50%. (This correlation between relative size of acceptance domains and frequencies is due to the minimal dimension of embedding).

Table II. — *Relative frequencies of occurrence of vertex configurations within the dodecagonal tiling* $\mathcal{P}^{(12)}$, $\varrho = 2 + \sqrt{3}$ *is the platinum number.*

| Vertex | Frequency of occurrence | | |
|---|---|---|---|
| 1 | $(15 - 4\varrho)/2$ | $\approx$ | 3.58984 % |
| 2 | $(-26 + 7\varrho)/2$ | $\approx$ | 6.21778 % |
| 3 | $3(4 - \varrho)/2$ | $\approx$ | 40.19238 % |
| 4 | $(4 - \varrho)/2$ | $\approx$ | 13.39746 % |
| 5 | $-26 + 7\varrho$ | $\approx$ | 12.43556 % |
| 6 | $(56 - 15\varrho)/2$ | $\approx$ | 0.96190 % |
| 7 | $(-26 + 7\varrho)/6$ | $\approx$ | 2.07259 % |
| 8 | $(-41 + 11\varrho)/2$ | $\approx$ | 2.62794 % |
| 9 | $(45 - 12\varrho)/2$ | $\approx$ | 10.76952 % |
| 10 | $(-26 + 7\varrho)/6$ | $\approx$ | 2.07259 % |
| 11 | $(19 - 5\varrho)/6$ | $\approx$ | 5.66253 % |

## 4. Concluding remarks.

For eight- and twelvefold symmetry, we presented a construction of relatively simple quasiperiodic tilings which possess perfect matching rules *inherently*, i.e., in such a way that these rules can be *locally* derived from the set of vertex sites alone. The concept of mutual local derivability was central in the chain of arguments, giving simultaneously the relation to other tilings that are already known and the method to construct the acceptance domains systematically.

We have not focussed explicitly on deflation/inflation properties of these tilings and their local nature. But this is another feature we get for free from the equivalence to the $D_4$-tilings [15]. Stated explicitly:

> The tilings $\mathcal{P}^{(8)}$ and $\mathcal{P}^{(12)}$ both possess *local* deflation and inflation property, i.e. the deflated as well as the inflated tilings are locally derivable from the original ones.

A discussion of that will be given in the Appendix.

Constructively, we know that local deflation/inflation symmetry and locally derivable matching rules are compatible with 8-, 10-, and 12-fold symmetry (cf. [10] for the 10-fold case). It is possible to construct simple discrete point set representatives. Now, one obvious question is whether this is an accidental coincidence linked to quadratic irrationalities. Preliminary investigations of tilings with 7-fold symmetry [22] and several general results on arbitrary symmetry groups and their relation to algebraic integers [14] indicate that this is not the case: one may expect the tripel "quasiperiodicity – local deflation/inflation – perfect matching rules" to be much more pertinent than it was expected so far, although we admit that the consideration of suitably closed tiling ensembles might be necessary.

**Acknowledgements.**

This article is part of a study on the theory of quasicrystals, performed at the Institute of Theoretical Physics, Tübingen University. It is a pleasure to thank P. Kramer for helpful advice and continuous support, M. Schlottmann for carefully reading the manuscript and suggesting various improvements, D. Joseph for providing important material and figure 1, and — last but

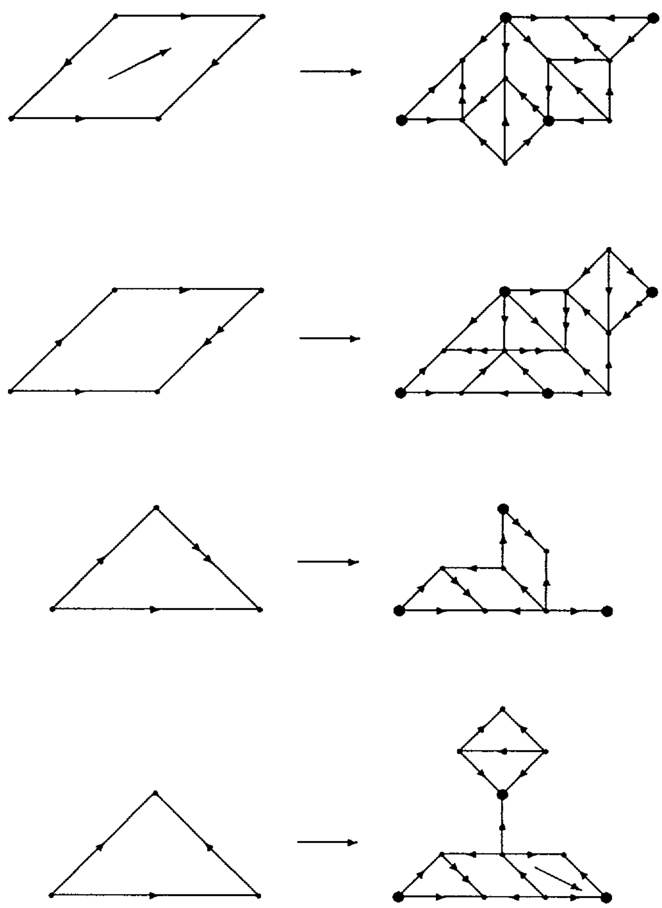

Fig. 11. — Local substitution rule for $\mathcal{P}^{(8)}$. It can be shown that the decoration used is locally derivable from the bare tiling. For $\mathcal{P}^{(12)}$ see text.

not least — F. Gähler and P.A.B. Pleasants for useful discussions. This work was supported by Deutsche Forschungsgemeinschaft.

## Appendix.

Let us comment on how the global inflation operates on the set of vertex sites, which in turn can be read from the vertex acceptance domains.

It has been shown that all these domains are star-shaped with respect to their centers. (A set is called star-shaped if it contains a point such that each line from this point to any other point of the set lies completely inside the set). The centers, however, are the possible fixed points of inflation/deflation transformations. Consequently, a global inflation of these patterns, corresponding to a shrink of the domains towards the fixed points — up to a permutation between congruent domains — yields smaller domains lying completely inside the original ones, i.e., considering the physical space of the tilings, all vertex sites of the inflated tilings coincide with some sites of the original tiling.

Now, let us give a brief discussion of the *local* (de-)composition rules, i.e. the substitutional aspect of the tilings $\mathcal{P}^{(8)}/\mathcal{P}^{(12)}$. In reference [10] the local (de-)composition rules have been shown for the tiles of $\mathcal{T}_{D_4}^{(8)}$ and $\mathcal{T}_{D_4}^{(12)}$. Hence, starting with some patch in $\mathcal{P}^{(8)}/\mathcal{P}^{(12)}$, one applies the local derivation rule towards $\mathcal{T}_{D_4}^{(8)}/\mathcal{T}_{D_4}^{(12)}$ first, then the (de-)composition rule there, and finally the local derivation rule back again.

Because of being tile to tile, the derivation rule yields, in the case of $\mathcal{P}^{(12)}$, a rather trivial transformation of the substitution rules of $\mathcal{T}_{D_4}^{(12)}$ given in reference [10]: The triangle which corresponds to the small triangle of $\mathcal{T}_{D_4}^{(12)}$decomposes into the shield; the other one which corresponds to the acute triangle of $\mathcal{T}_{D_4}^{(12)}$decomposes into two shields, two triangles of the same kind and an additional rhombus; the shield decomposes into one triangle of the first kind and three of the latter; and, due to the fact that this decomposition rule is not shape preserving, the rhombus will vanish.

An analogous procedure for the 8-fold case results, if formulated as simple as possible, in a local decomposition rule for *two* locally distinguishable kinds of oriented rhombi and *two* (again distinguishable) rectangular triangles, as shown in figure 11. Therefore, $\mathcal{P}^{(8)}$ yields a four tile substitution rule. Thereby, the squares are to be devided along the orientating arrows, introduced in section 2. By the way, that arrow stems from the decoration of this dividing line. Thus, part of the proof for the locality of the decoration introduced here has already been given. What remains is an as easy exercise.

Clearly, in the 8-fold case the deflation factor is $\lambda$; in the 12-fold it is $\sqrt{\varrho}$.

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