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LETTER TO THE EDITOR

Quasiperiodic icosahedral tilings from the six-dimensional bcc lattice

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Abstract. The cell geometry of the six-dimensional bcc lattice is investigated. Via klotz construction two different classes of icosahedrally projected quasiperiodic tilings are defined. For both cases we determine the acceptance domains of tiles and give a detailed description of the geometry of all tiles.

1. Introduction

As has been shown by Rokhsar *et al* [1], there exist only three icosahedral modules (in $\mathbb{R}^3$) of rank 6. They can be obtained by icosahedral projection from the six-dimensional (6D) primitive cubic lattice P, i.e. $\mathbb{Z}^6$, the face-centred cubic lattice $2F$, i.e. the root lattice D_6, and the body-centred cubic lattice I (reciprocal to $2F$), i.e. the weight lattice D_6^R, respectively. The icosahedral projection from 6D to 3D space is defined by a particular embedding, $[31^2_+]$, of the 3D faithful representation of the symmetry group, Y_h, of the icosahedron in the 6D representation of the higher-dimensional (6D) lattice, $\mathbb{Z}^6$, D_6 or D_6^R, see [2–4]. The 6D space splits as $\mathbb{E}^6 = \mathbb{E}_\parallel \oplus \mathbb{E}_\perp$, where $\mathbb{E}_\parallel$ is the representation space of $[31^2_+]$, the (physical) space of the quasiperiodic tiling, and $\mathbb{E}_\perp$ is the representation space of $[31^2_-]$, the (internal) space of the coding [3, 5]. In the projection procedure from the 6D lattice we define two local isomorphism (LI) classes of tilings [3, 6], $\mathcal{T}$ and $\mathcal{T}^\star$: the tiles of the LI class $\mathcal{T}$ in $\mathbb{E}_\parallel$ are icosahedrally projected 3D boundaries of the Voronoi cell $P_\parallel(3)$ and are coded by icosahedrally projected dual boundaries $P_\perp^\star(3)$ within $\mathbb{E}_\perp$, cf [5]; the tiles of the LI class $\mathcal{T}^\star$ are the icosahedrally projected 3D boundaries of the Delaunay cells $P_\parallel^\star(3)$, coded by $P_\perp(3)$. Note that the tilings $\mathcal{T}$ and $\mathcal{T}^\star$ coincide only in the case of $\mathbb{Z}^6$. Quasiperiodic tilings obtained by icosahedral projection from $\mathbb{Z}^6$ and from D_6 have been studied extensively [2–4, 7–9].

2. To the tiles and tilings $\mathcal{T}^{(\mathbf{I})}$ and $\mathcal{T}^{\star(\mathbf{I})}$

We now consider quasiperiodic tilings obtained by icosahedral projection from the weight lattice D_6^R. By various methods [10, 11] we have determined, in 6D, the hierarchy of boundaries of the Voronoi cell, a polytope with Schläfli symbol $\{{}^{33}_{334}\}$, and of the Delaunay cells, one representative of which being the convex hull of the 16 points $\{\frac{1}{2}(\pm1, \pm1, \pm1, 0, 0, 0)\} \cup \{\frac{1}{2}(0, 0, 0, \pm1, \pm1, \pm1)\}$, more details can be found in table 1. Here we only describe the 3D boundaries $P(3)$ and $P^\star(3)$. The 3D boundaries of the

Table 1. The incidence matrices of the 6D topology for the Voronoi cell, V, (above) and one representative Delaunay cell, D, (below). Entries N_{ij} are to be read as follows: each i-boundary coincides with N_{ij} j-boundaries; N_{ii} counts the total number of i-boundaries. The boundaries are subdivided into different orbits with respect to the pointgroup.

V	0D	1D	2D		3D			4D			5D	
0D	160	18	36	8	24	6	36	3	24	12	8	3
1D	2	1440	4	2	4	1	8	1	8	4	4	2
2D	3	3	1920	—	2	0	2	1	4	1	2	2
	3	3	—	960	0	1	4	0	4	4	4	1
3D	4	6	4	0	960	—	—	1	2	0	1	2
	4	6	0	4	—	240	—	0	0	4	4	0
	6	12	4	4	—	—	960	0	2	1	2	1
4D	8	24	32	0	16	0	0	60	—	—	0	2
	10	30	20	10	5	0	5	—	384	—	1	1
	10	30	10	20	0	5	5	—	—	192	2	0
5D	20	90	60	60	15	15	30	0	6	6	64	—
	40	240	320	80	160	0	80	10	32	0	—	12

D	0D		1D			2D				3D			4D		5D
0D	8	—	3	0	8	24	12	3	0	6	24	36	18	36	18
	—	8	0	3	8	12	24	0	3	24	6	36	36	18	18
1D	2	0	12	—	—	8	0	2	0	0	16	12	6	24	12
	0	2	—	12	—	0	8	0	2	16	0	12	24	6	12
	1	1	—	—	64	3	3	0	0	3	3	9	9	9	9
2D	2	1	1	0	2	96	—	—	—	0	2	3	3	6	6
	1	2	0	1	2	—	96	—	—	2	0	3	6	3	6
	4	0	4	0	0	—	—	6	—	0	8	0	0	12	6
	0	4	0	4	0	—	—	—	6	8	0	0	12	0	6
3D	1	4	0	4	4	0	4	0	1	48	—	—	3	0	3
	4	1	4	0	4	4	0	1	0	—	48	—	0	3	3
	2	2	1	1	4	2	2	0	0	—	—	144	2	2	4
4D	2	4	1	4	8	4	8	0	1	2	0	4	72	—	2
	4	2	4	1	8	8	4	1	0	0	2	4	—	72	2
5D	4	4	4	4	16	16	16	1	1	4	4	16	4	4	36

Voronoi cell, $P(3) \subset V(0)$, are 1200 tetrahedra (T) and 960 octahedra (O), all with edges of the same length $1/\sqrt{2}$ (scaled such that the primitive basis of $\mathbb{Z}^6$, e_i, $i = 1, \ldots, 6$, obeys $(e_i, e_j) = \delta_{ij}$). There are 10 congruent Delaunay cells, $D^{(j)}$, $j = 1, \ldots, 10$. Each one has, as 3D boundaries $P^\star(3) \subset D^{(j)}$, 96 pyramids $T^\star$ and 144 tetrahedra $O^\star$. Each pyramid $T^\star$ has, as a base, the square of edge length 1, the lateral edges have length $\sqrt{3/2}$.

The icosahedrally projected Delaunay cells $D_\perp^{(j)}$ are the vertex windows or acceptance domains for the tilings in the LI class $\mathcal{T}^{(I)}$. $D_\perp^{(j)}$ has the shape of the scalenohedron with the symmetry D_{3v} (see figure 1) and the class $\mathcal{T}^{(I)}$ does not contain a non-singular tiling with global icosahedral symmetry†. Moreover, it does not even contain a non-singular tiling with global D_{3v} symmetry. The tiles of $\mathcal{T}^{(I)}$ are $P_\|(3)$, i.e. icosahedrally projected tetrahedra $T_\|$ and octahedra $O_\|$. The tetrahedra $T_\|$ show five forms. One of them is degenerate, which we can simply remove here because they are not needed for the cell construction [5]. The other four, $T_{i\|}, i = 1, \ldots, 4$, coincide with the tiles $A_\|^\star$, $B_\|^\star$, $C_\|^\star$, and $D_\|^\star$ of the

† Even a singular one is impossible if the 10 translation classes are distinguished, as the group that generate $D_\perp^{(j)}$ is the Weyl group of the diagram [11] $A_3 \times A_3$, so does not allow the embedding of Y_h. On the other hand, no tile has icosahedral symmetry.

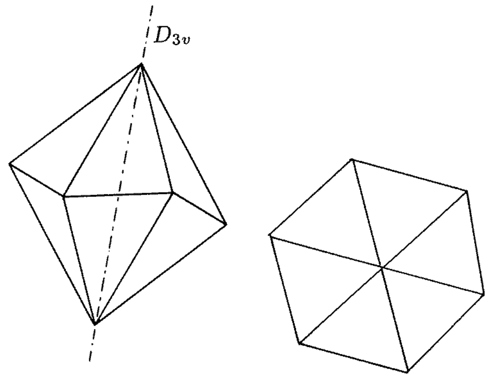

Figure 1. The Delaunay cell $D_\perp^{(j)}$ in two board projection.

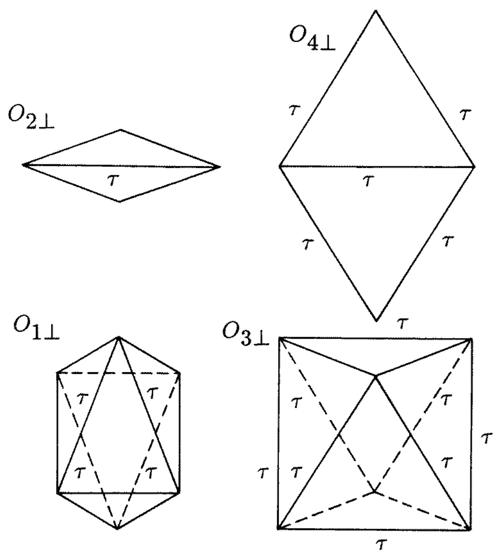

Figure 2. The tiles of $\mathcal{T}^{(I)}$, $O_{i\perp}$ $(i = 1, \ldots, 4)$, in an orthogonal projection.

tiling [3] $\mathcal{T}^{\star(2F)}$ (scaled by a factor $\frac{1}{2}$). The octahedra $O_\parallel$ appear in five forms, again one degenerate; the other $O_{i\parallel}$, $i = 1, \ldots, 4$ are shown in figure 2. The shapes of them are all double pyramids, point symmetric with respect to the centre of the base. All edges are parallel to 2-fold symmetry axes of an icosahedron, and only two different edge lengths occur, $\boxed{2} = \frac{1}{2}\sqrt{2/(\tau+2)}$ and $\tau \cdot \boxed{2}$, τ the golden ratio†. The generating pyramids of $O_{2\parallel}$ and $O_{4\parallel}$ have rectangular bases and small/long lateral edges, respectively; those of $O_{1\parallel}$ and $O_{3\parallel}$ are oblique and based on a small/big square, respectively.

The icosahedrally projected Voronoi cell $V_\perp(0)$ forms a dodecahedron with edge length $\tau \cdot \boxed{2}$. It is the vertex window (or acceptance domain) of the LI class $\mathcal{T}^{\star(I)}$. This class

† Note that the smallest inflation factor of the icosahedrally projected D_6^R is τ, just as for D_6.

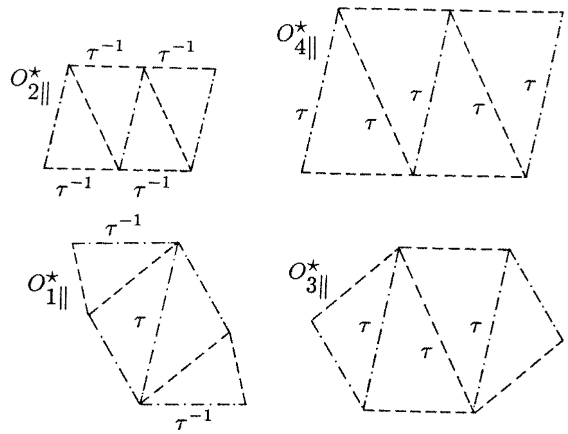

Figure 3. The unfolded tetrahedra $O^\star_{i\|}$, $i = 1, \ldots, 4$.

contains (up to translations) one tiling with global icosahedral symmetry. The tiles are four non-degenerate pyramids $T^\star_{i\|}$, $i = 1, \ldots, 4$, coinciding with four out of the six tiles [4] of $\mathcal{T}^{(2F)}$, and, in addition, four non-degenerate tetrahedra, $O^\star_{i\|}$, $i = 1, \ldots, 4$ (the latter are shown in figure 3). All edges (of $T^\star_{i\|}$ and $O^\star_{i\|}, i = 1, \ldots, 4$) are either parallel to 3-fold directions of an icosahedron (— · —) with two different edge lengths $③ = \frac{1}{2}\sqrt{6/(\tau+2)}$ and $\tau \cdot ③$, or parallel to 5-fold directions (– – –) with three different edge lengths $⑤ = 1/\sqrt{2}$, $\tau^{-1} \cdot ⑤$ and $\tau \cdot ⑤$. Within figure 3, scalings by powers of τ with respect to a standard length ⑤ and ③ are marked.

3. Conclusion

The icosahedral quasicrystals related to the P- and $2F$-module, icosahedrally projected from the $\mathbb{Z}^6$ and D_6 lattice, respectively, have been experimentally observed (see for example [12, 13]). No quasicrystals related to the I-module, projected from the D_6^R lattice have been observed so far. Nevertheless, the above introduced new classes of tilings $\mathcal{T}^{(I)}$ and $\mathcal{T}^{\star(I)}$ are also of interest for further investigations as mathematical structures.

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