LETTER TO THE EDITOR

Quasiperiodic icosahedral tilings from the six-dimensional bcc lattice

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Abstract. The cell geometry of the six-dimensional bcc lattice is investigated. Via klotz construction two different classes of icosahedrally projected quasiperiodic tilings are defined. For both cases we determine the acceptance domains of tiles and give a detailed description of the geometry of all tiles.

1. Introduction

As has been shown by Rokhsar et al [1], there exist only three icosahedral modules (in \(\mathbb{R}^3\)) of rank 6. They can be obtained by icosahedral projection from the six-dimensional (6D) primitive cubic lattice \(P\), i.e. \(Z^6\), the face-centred cubic lattice \(2F\), i.e. the root lattice \(D_6\), and the body-centred cubic lattice \(I\) (reciprocal to \(2F\)), i.e. the weight lattice \(DR_6\), respectively. The icosahedral projection from 6D to 3D space is defined by a particular embedding, \([312_+^+]\), of the 3D faithful representation of the symmetry group, \(Y_h\), of the icosahedron in the 6D representation of the higher-dimensional (6D) lattice, \(Z^6\), \(D_6\) or \(DR_6\), see [2–4]. The 6D space splits as \(E^6 = E^\parallel \oplus E^\perp\), where \(E^\parallel\) is the representation space of \([312^+]\), the (physical) space of the quasiperiodic tiling, and \(E^\perp\) is the representation space of \([312^-]\), the (internal) space of the coding [3, 5]. In the projection procedure from the 6D lattice we define two local isomorphism (LI) classes of tilings [3, 6], \(T\) and \(T^*\): the tiles of the LI class \(T\) in \(E^\parallel\) are icosahedrally projected 3D boundaries of the Voronoi cell \(P_1(3)\) and are coded by icosahedrally projected dual boundaries \(P^*_1(3)\) within \(E^\perp\), cf [5]; the tiles of the LI class \(T^*\) are the icosahedrally projected 3D boundaries of the Delaunay cells \(P^*_1(3)\), coded by \(P_1(3)\). Note that the tilings \(T\) and \(T^*\) coincide only in the case of \(Z^6\). Quasiperiodic tilings obtained by icosahedral projection from \(Z^6\) and from \(D_6\) have been studied extensively [2–4, 7–9].

2. To the tiles and tilings \(T^{(I)}\) and \(T^{*(I)}\)

We now consider quasiperiodic tilings obtained by icosahedral projection from the weight lattice \(D_6^K\). By various methods [10, 11] we have determined, in 6D, the hierarchy of boundaries of the Voronoi cell, a polytope with Schlaffi symbol \(\{13\}_{333}\), and of the Delaunay cells, one representative of which being the convex hull of the 16 points \(\{0, 0, \pm 1, \pm 1, 0, 0\} \cup \{\pm 1, \pm 1, \pm 1, 0, 0\}\), more details can be found in table 1. Here we only describe the 3D boundaries \(P(3)\) and \(P^*(3)\). The 3D boundaries of the
Table 1. The incidence matrices of the 6D topology for the Voronoi cell, $V$, (above) and one representative Delaunay cell, $D$, (below). Entries $N_{ij}$ are to be read as follows: each $i$-boundary coincides with $N_{ij}$ $j$-boundaries; $N_{ii}$ counts the total number of $i$-boundaries. The boundaries are subdivided into different orbits with respect to the pointgroup.

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<th>3D</th>
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<tr>
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<td>32</td>
<td>0</td>
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Voronoi cell, $P(3) \subset V(0)$, are 1200 tetrahedra ($T$) and 960 octahedra ($O$), all with edges of the same length $1/\sqrt{2}$ (scaled such that the primitive basis of $Z^6$, $e_i$, $i = 1, \ldots , 6$, obeys $(e_i, e_j) = \delta_{ij}$). There are 10 congruent Delaunay cells, $D^{(j)}$, $j = 1, \ldots , 10$. Each one has, as 3D boundaries $P^*(3) \subset D^{(j)}$, 96 pyramids $T^*$ and 144 tetrahedra $O^*$. Each pyramid $T^*$ has, as a base, the square of edge length 1, the lateral edges have length $\sqrt{3}/2$.

The icosahedrally projected Delaunay cells $D^{(j)}_\perp$ are the vertex windows or acceptance domains for the tilings in the LI class $T^{(j)}$. $D^{(j)}_\perp$ has the shape of the scalenohedron with the symmetry $D_{3h}$ (see figure 1) and the class $T^{(j)}_\perp$ does not contain a non-singular tiling with global icosahedral symmetry†. Moreover, it does not even contain a non-singular tiling with global $D_{3h}$ symmetry. The tiles of $T^{(j)}_\perp$ are $P(3)$, i.e. icosahedrally projected tetrahedra $T^*_\perp$ and octahedra $O^*_\perp$. The tetrahedra $T^*_\perp$ show five forms. One of them is degenerate, which we can simply remove here because they are not needed for the cell construction [5]. The other four, $T^*_\parallel$, $i = 1, \ldots , 4$, coincide with the tiles $A^*_\parallel$, $B^*_\parallel$, $C^*_\parallel$, and $D^*_\parallel$ of the

† Even a singular one is impossible if the 10 translation classes are distinguished, as the group that generate $D^{(j)}_\perp$ is the Weyl group of the diagram [11] $A_3 \times A_3$, so does not allow the embedding of $Y_5$. On the other hand, no tile has icosahedral symmetry.
Figure 1. The Delaunay cell $D_{\parallel}^{(1)}$ in two board projection.

Figure 2. The tiles of $T(I)$, $O_i\perp (i=1,...,4)$, in an orthogonal projection.

tiling [3] $T^{*(2F)}$ (scaled by a factor $\frac{1}{2}$). The octahedra $O_i\parallel$ appear in five forms, again one degenerate; the other $O_i\parallel$, $i=1,...,4$ are shown in figure 2. The shapes of them are all double pyramids, point symmetric with respect to the centre of the base. All edges are parallel to 2-fold symmetry axes of an icosahedron, and only two different edge lengths occur, $2 = \frac{1}{2}\sqrt{2/(\tau + 2)}$ and $\tau \cdot 2$, $\tau$ the golden ratio.$^\dagger$. The generating pyramids of $O_2\parallel$ and $O_4\parallel$ have rectangular bases and small/long lateral edges, respectively; those of $O_1\parallel$ and $O_3\parallel$ are oblique and based on a small/big square, respectively.

The icosahedrally projected Voronoi cell $V_{\perp}(0)$ forms a dodecahedron with edge length $\tau \cdot 2$. It is the vertex window (or acceptance domain) of the LI class $T^{*(I)}$. This class

$^\dagger$ Note that the smallest inflation factor of the icosahedrally projected $D_6^R$ is $\tau$, just as for $D_6$. 
Figure 3. The unfolded tetrahedra $O^*_{i\parallel}, i = 1, \ldots, 4$.

contains (up to translations) one tiling with global icosahedral symmetry. The tiles are four non-degenerate pyramids $T^*_{i\parallel}, i = 1, \ldots, 4$, coinciding with four out of the six tiles [4] of $T^{(2F)}$, and, in addition, four non-degenerate tetrahedra, $O^*_{i\parallel}, i = 1, \ldots, 4$ (the latter are shown in figure 3). All edges (of $T^*_{i\parallel}$ and $O^*_{i\parallel}, i = 1, \ldots, 4$) are either parallel to 3-fold directions of an icosahedron (—•—) with two different edge lengths $\tau \cdot \tau^3 = 1/2 \sqrt{6/(\tau + 2)}$ and $\tau \cdot 3$, or parallel to 5-fold directions (———) with three different edge lengths $5 = 1/\sqrt{2}, \tau^{-1} \cdot 5$ and $\tau \cdot 5$. Within figure 3, scalings by powers of $\tau$ with respect to a standard length $5$ and $3$ are marked.

3. Conclusion

The icosahedral quasicrystals related to the $P$- and $2F$-module, icosahedrally projected from the $Z_6$ and $D_6$ lattice, respectively, have been experimentally observed (see for example [12, 13]). No quasicrystals related to the $I$-module, projected from the $D_{12}^R$ lattice have been observed so far. Nevertheless, the above introduced new classes of tilings $T(I)$ and $T^*(I)$ are also of interest for further investigations as mathematical structures.

References

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