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AXIAL-SYMMETRICAL EDGE-FACETINGS OF UNIFORM POLYHEDRA

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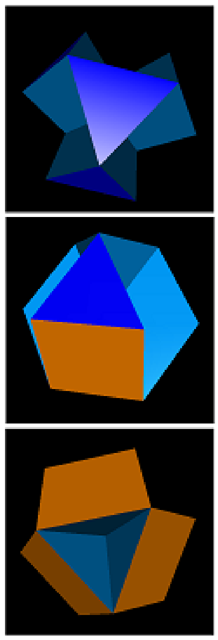
Abstract: The set of uniform polyhedra is grouped into classes of figures with the same edge skeletons. For each class one representative is chosen. Each such class is investigated further for other polyhedra with regular faces only, following the constraint that the edges remain a non-empty subset of the skeleton under consideration.

INTRODUCTION

In the days of the old Greeks Plato enumerated the five convex regular polyhedra which are henceforth associated with his name. A further set of 13 convex polyhedra was named after Archimedes, these have regular faces only, of at least 2 different kinds, and the vertex figures of the polyhedra being transitively permuted by some symmetry which has more than just one "true" rotational axis. I.e. the infinite set of prisms and antiprisms is classically not subsumed under his name.

After that early period of polyhedral interest it took some time up to the days of Kepler and Poinsot who firstly enumerated the additional 4 non-convex regular polyhedra.



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The ones which are stellations of the convex ones were found first, while the ones which are facetings followed a bit later. But it was only in the last century that a team around Coxeter managed to enumerate an still larger (inclusive) set of polyhedra which extends the set of 9 regular polyhedra in the same way that the Archimedeans (taken in an inclusive sense, and subsuming the 2 prism series as well) are related to the Platonics.

This set of figures was called the set of uniform polyhedra. Those are uniform in the sense of having just one type of vertex figure, which is spread throughout transitively by the action of some symmetry group. A second rule understood by "uniform polyhedra" is that all edges have to be of equal length, or, what comes to be the same but is better suited for dimensional extension, is that this second part of uniformity has to be applied as a dimensional recursion. Hereby starting the iteration with uniform polygons considered to be the same as regular ones. Then generally polytopes are considered uniform if vertex transitivity applies and secondly all facets are uniform in turn.

Table of uniform polyhedra:

No	Name of uniform polyhedron	Bowers acronym	Dynkin symbol *)	Wythoff symbol	Sequence of faces per vertex **)
U01	Tetrahedron	tet	@-3-o-3-o	3 | 2 3	[3^3]
U02	Truncated tetrahedron	tut	@-3-@-3-o	2 3 | 3	[3,6^2]
U03	Octahemioctahedron	oho	@-3-@-3/2-o-3-:	3/2 3 | 3	[3/2,6,3,6]
U04	Tetrahomihexahedron	thah	@-3/2-o-3-@	3/2 3 | 2	[3/2,4,3,4]
U05	Octahedron, tetratetrahedron	oct	o-3-@-3-o, @-3-o-4-o	2 | 3 3, 4 | 2 3	[3^4]
U06	Cube, hexahedron	cube	@-4-o-3-o	3 | 2 4	[4^3]
U07	Cuboctahedron, (small) rhombitetratetrahedron	co	@-3-o-3-@, o-3-@-4-o	3 3 | 2, 2 | 3 4	[(3,4)^2]
U08	Truncated octahedron, (omni-)truncated tetratetrahedron, great rhombitetratetrahedron	toe	@-3-@-3-@, @-3-@-4-o	2 3 3 |, 2 4 | 3	[4,6^2]
U09	Truncated cube	tic	@-4-@-3-o	2 3 | 4	[3,8^2]
U10	(Small) Rhombicuboctahedron	sirco	@-3-o-4-@	3 4 | 2	[3,4^3]
U11	(Omni-)truncated cuboctahedron, Great rhombicuboctahedron	girco	@-3-@-4-@	2 3 4 |	[4,6,8]
U12	Snub cube	snic	O-3-O-4-O	| 2 3 4	[3^4,4]
U13	Small cubicuboctahedron	socco	@-4-@-3/2-o-4-:	3/2 4 | 4	[3/2,8,4,8]
U14	Great cubicuboctahedron	gocco	@-4/3-@-3-o-4-:	3 4 | 4/3	[8/3,3,8/3,4]
U15	Cubohemioctahedron	cho	@-3-@-4/3-o-4-:	4/3 4 | 3	[4/3,6,4,6]
U16	Cubitruncated cuboctahedron, cuboctatruncated cuboctahedron	cotco	@-4/3-@-3-@-4-:	4/3 3 4 |	[8/3,6,8]
U17	Great rhombicuboctahedron, quasirhombicuboctahedron	querco	@-3/2-o-4-@	3/2 4 | 2	[3/2,4^3]
U18	Small rhombihexahedron	sroh	@-3/2-@-4-@ ***)	3/2 2 4 | ***)	[8/7,4/3,8,4]

No	Name of uniform polyhedron	Bowers acronym	Dynkin symbol *)	Wythoff symbol	Sequence of faces per vertex **)
U19	Stellated truncated hexahedron, quasitruncated hexahedron	quith	@-4/3-@-3-o	2 3 | 4/3	[(8/3)^2,3]
U20	Great truncated cuboctahedron, quasitruncated cuboctahedron	quitco	@-4/3-@-3-@	4/3 2 3 |	[8/3,4,6]
U21	Great rhombihexahedron	groh	@-4/3-@-3/2-@ ***)	4/3 3/2 2 | ***)	[4/3,8/5,4,8/3]
U22	Icosahedron, Snub tetrahedron	ike	O-3-O-3-O, @-3-o-5-o	2 3 3 |, 5 | 2 3	[3^5]
U23	Dodecahedron	doe	@-5-o-3-o	3 | 2 5	[5^3]
U24	Icosidodecahedron	id	o-3-@-5-o	2 | 3 5	[(3,5)^2]
U25	Truncated icosahedron, “soccer ball”	ti	@-3-@-5-o	2 5 | 3	[5,6^2]
U26	Truncated dodecahedron	tid	@-5-@-3-o	2 3 | 5	[3,10^2]
U27	(Small) Rhombicosidodecahedron	srid	@-3-o-5-@	3 5 | 2	[3,4,5,4]
U28	(Omni-)truncated icosidodecahedron, great rhombicosidodecahedron	grid	@-3-@-5-@	2 3 5 |	[4,6,10]
U29	Snub dodecahedron	snid	O-3-O-5-O	| 2 3 5	[3^4,5]
U30	Small ditrigonal icosidodecahedron	sidtid	o-5/2-@-3-o-3-:	3 | 5/2 3	[(5/2,3)^3]
U31	Small icosicosidodecahedron	siid	@-3-@-5/2-o-3-:	5/2 3 | 3	[5/2,6,3,6]
U32	Small snub icosicosidodecahedron	seside	O-5/2-O-3-O-3-:	| 5/2 3 3	[5/2,3^5]
U33	Small dodekicosidodecahedron	saddid	@-5-@-3/2-o-5-:	3/2 5 | 5	[3/2,10,5,10]
U34	Small stellated dodecahedron	sissid	@-5/2-o-5-o	5 | 2 5/2	[(5/2)^5]
U35	Great dodecahedron	gad	@-5-o-5/2-o	5/2 | 2 5	[5^5]/2
U36	Dodecadodecahedron	did	o-5/2-@-5-o	2 | 5/2 5	[(5/2,5)^2]
U37	Truncated great dodecahedron	tigid	@-5-@-5/2-o	2 5/2 | 5	[5/2,10^2]
U38	Rhombidodecadodecahedron	raded	@-5/2-o-5-@	5/2 5 | 2	[5/2,4,5,4]
U39	Small rhombidodecadodecahedron	sird	@-5/2-@-5-@ ***)	2 5/2 5 | ***)	[10/9,4/3,10,4]
U40	Snub dodecadodecahedron	siddid	O-5/2-O-5-O	| 2 5/2 5	[5/2,3^2,5,3]
U41	Ditrigonal dodecadodecahedron	ditdid	@-5/3-o-3-o-5-:	3 | 5/3 5	[(5/3,5)^3]
U42	Great ditrigonal dodekicosidodecahedron	gidditdid	@-5/3-@-3-o-5-:	3 5 | 5/3	[3,10/3,5,10/3]
U43	Small ditrigonal dodekicosidodecahedron	sidditdid	@-5-@-5/3-o-3-:	5/3 3 | 5	[5/3,10,3,10]
U44	Icosidodecadodecahedron	ided	@-3-@-5/3-o-5-:	5/3 5 | 3	[5/3,6,5,6]
U45	Icositruncated dodecadodecahedron, icosidodecatruncated icosidodecahedron	idtid	@-5/3-@-3-@-5-:	5/3 3 5 |	[10/3,6,10]
U46	Snub icosidodecadodecahedron	sided	O-5/3-O-3-O-5-:	| 5/3 3 5	[5/3,3^3,5,3]
U47	Great ditrigonal icosidodecahedron	gidtid	o-3-@-5-o-3/2-:	3/2 | 3 5	[(3,5)^3]/2
U48	Great icosicosidodecahedron	giid	@-3-@-3/2-o-5-:	3/2 5 | 3	[5/2,6,5,6]
U49	Small icosihemidodecahedron	seihid	@-5-@-3/2-o-3-:	3/2 3 | 5	[3/2,10,3,10]
U50	Small dodekicosahedron	siddy	@-3/2-@-3-@-5-: ***)	3/2 3 5 | ***)	[10/9,6/5,10,6]
U51	Small dodecahemidodecahedron	sidhid	@-5-@-5/4-o-5-:	5/4 5 | 5	[5/4,10,5,10]
U52	Great stellated dodecahedron	gissid	@-5/2-o-3-o	3 | 2 5/2	[(5/2)^3]
U53	Great icosahedron	gike	@-3-o-5/2-o	5/2 | 2 3	[3^5]/2
U54	Great icosidodecahedron	gid	o-5/2-@-3-o	2 | 5/2 3	[(5/2,3)^2]
U55	Great truncated icosahedron, truncated great icosahedron	tiggy	@-3-@-5/2-o	2 5/2 | 3	[5/2,6^2]

No	Name of uniform polyhedron	Bowers acronym	Dynkin symbol *)	Wythoff symbol	Sequence of faces per vertex **)
U56	Rhombicosahedron	ri	@-5/2-@-3-@ ***)	2 5/2 3 | ***)	[6/5,4/3,6,4]
U57	Great snub icosidodecahedron	gosid	O-5/2-O-3-O	| 2 5/2 3	[5/2,3^4]
U58	Small stellated truncated dodecahedron, quasitruncated small stellated dodecahedron	quit sissid	@-5/3-@-5-o	2 5 | 5/3	[(10/3)^2,5]
U59	(Quasi-)truncated dodecadodecahedron	quitdid	@-5/3-@-5-@	5/3 2 5 |	[10/3,4,10]
U60	Inverted snub dodecadodecahedron	isdid	O-5/3-O-5-O	| 5/3 2 5	[5/3,3^2,5,3]
U61	Great dodekicosidodecahedron	gaddid	@-5/3-@-5/2-o-3-:	5/2 3 | 5/3	[5/2,10/3,3,10/3]
U62	Small dodecahemicosahedron	sidhei	@-3-@-5/3-o-5/2-:	5/3 5/2 | 3	[5/3,6,5/2,6]
U63	Great dodekicosahedron	giddy	@-5/3-@-5/2-@-5-: ***)	5/3 5/2 3 | ***)	[6/5,10/7,6,10/3]
U64	Great snub dodekicosidodecahedron	gisdid	O-5/3-O-5/2-O-5-:	| 5/3 5/2 3	[5/3,3^3,5/2,3]
U65	Great dodecahemicosahedron	gidhei	@-3-@-5/4-5-:	5/4 5 | 3	[5/4,6,5,6]
U66	Great stellated truncated dodecahedron, quasitruncated great stellated dodecahedron	quit gissid	@-5/3-@-3-o	2 3 | 5/3	[(10/3)^2,3]
U67	Great rhombicosidodecahedron, quasirhombicosidodecahedron	qrid	@-5/3-o-3-@	5/3 3 | 2	[5/3,4,3,4]
U68	Great (quasi-)truncated icosidodecahedron	gaquatid	@-5/3-@-3-@	5/3 2 3 |	[10/3,4,6]
U69	Great inverted snub icosidodecahedron	gisid	O-5/3-O-3-O	| 5/3 2 3	[5/3,3^4]
U70	Great dodecahemidodecahedron	gidhid	@-5/3-@-5/3-o-5/2-:	5/3 5/2 | 5/3	[(10/3,5/3)^2]
U71	Great icosihemidodecahedron	geihid	@-5/3-@-3/2-o-3-:	3/2 3 | 5/3	[3/2,10/3,3,10/3]
U72	Small (inverted) retrosnub icosicosidodecahedron, yog sothoth	sirsid	O-3/2-O-3/2-O-5/2-:	| 3/2 3/2 5/2	[(3/2,3)^2,5/2,3]
U73	Great rhombidodecahedron	gird	@-3/2-@-5/3-@ ***)	3/2 5/3 2 | ***)	[4/3,10/7,4,10/3]
U74	Great retrosnub icosidodecahedron	girsid	O-3/2-O-5/3-O	| 3/2 5/3 2	[3/2,3,5/3,3^2]
U75	Great dirhombicosidodecahedron, Miller's monster	gidrid	None	| 3/2 5/3 3 5/2	[3/2,4,5/3,4,3,4,5/2,4]
****)	Pentagonal prism	pip	@ @-5-o	2 5 | 2	[4^2,5]
****)	Pentagonal antiprism	pap	O O-5-O	| 2 2 5	[3^3,5]
****)	Pentagrammic prism, stellar prism	stip	@ @-5/2-o	5/2 2 | 2	[5/2,4^2]
****)	Pentagrammic antiprism, stellar antiprism	stap	O O-5/2-O	| 5/2 2 2	[5/2,3^3]
****)	Pentagrammic crossed antiprism, pentagrammic retrograde antiprism, pentagrammic retroprism, stellar retroprism	starp	O O-5/3-O	| 5/3 2 2	[5/3,3^3]

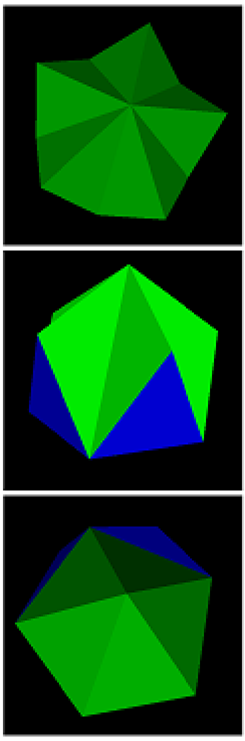
*) Circularly closed Dynkin graphs are for linearized denotation cut upon somewhere, a final colon reminds then to reconnect that final open link back to the first knot.

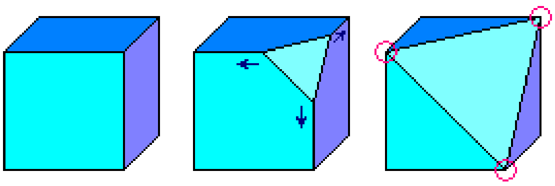
**) Exponents in the face sequence of vertex figures are to be read as multiplicities of the corresponding face or partial face sequence within the total sequence. Divisors behind the square bracket are to be read as winding numbers. (Winding numbers of 0 or 1 are suppressed.)

***) More precisely that symbol does specify a somehow exotic polyhedron with faces of the kind `@-n/d-@` with even divisor. Thus that face would have pairwise coincident vertices and edges. The uniform polyhedron, as usually understood, is a reduction of that exotic one. Those exotic faces are withdrawn, while the thus opened coincident edges are identified.
****) Entries beyond U75 just represent some examples from the infinite dihedral series.

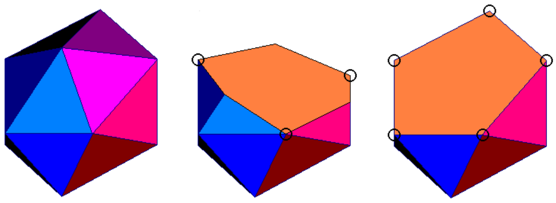
FACETINGS

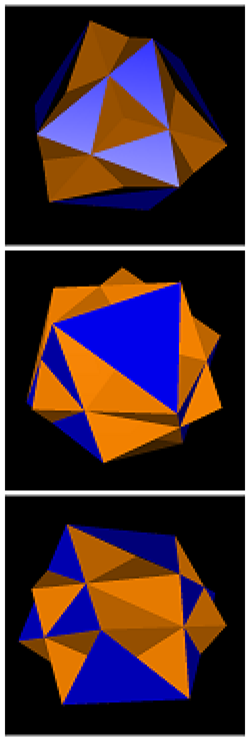
Truncation is a well-known process which can be applied to different more or less symmetric polyhedra. Many of the Archimedeans are derivable from the Platonics through truncation. Truncation introduces additional faces as cutting planes, while former vertices or edges are removed. That process of truncation can be considered continuous starting at one end of the polyhedron, running through it getting deeper and deeper, until the cutting plane passes through the opposite end, and the former polyhedron is reduced to the empty set. Application of that process of truncation simultaneously at symmetry equivalent places to some symmetrical figure increases the esthetical appeal.





Truncation vs. faceting





ike-5-5 General faceting vs. edge-faceting sidtid-0-8-12-0-b

Already the set of Archimedeans shows that special instances in that continuous process are of higher interest than others. Suppose now that the polygon, defined by intersection of the cutting plane with the polyhedron, has its vertices only at vertex positions of the former polyhedron. In that case one speaks of a faceting.

Note that facetings are dual to the more elaborated stellations, just as apiculations are dual to truncations, a process used in the dual set of the Archimedeans, the Catalans. A well known fact is that there are different types of stellations. First there are those which can be derived by extending the edges. For instance, extending the edges of the pentagons of the dodecahedron (doe), yields pentagrammic faces which reconnect to produce the small stellated dodecahedron (sissid). A looser type of stellation can be obtained by enlarging the faces again, but using face-plane intersections for new edges, which were not contained in the previously edge set. This latter kind of stellation is also known under the name greatening. For instance, by extending the faces of an octahedron (oct) its possible to get the compound of 2 tetrahedra (tet) positioned dually. Kepler introduced the name Stella Octangula for that compound. In the same way as there are 2 types of stellations, there are 2 types of facetings too. The looser type of stellation, the process of greatening, is dually associated to the process of faceting just introduced. It keeps all (or some) faces, whereas the old edges do not contribute to the set of the new ones. Similarly, all (or some) vertices are kept while faceting in the looser sense, and again the old edges do not contribute to the set of the new ones. Opposed to that, stellation in the stricter sense reuses both, face-planes and edges. And there is a dual process as well, the more restricted type of faceting, which reuses not only vertices but also the old edges. That stricter type of faceting is usually called edge-faceting and will be the main subject of this article.

Restricting to edge-facetings, it gets obvious too, that no polyhedra with just 3 faces at some vertex could have any faceting face incident to that vertex other than the 3 existing ones. So, many of the uniforms disqualify for edge-facetation a priori. As further uniform polyhedra just have edges of one length, edge-facetation can only produce new faceting faces with the same property.

The task of edge-faceting the set of uniform polyhedra while keeping their own symmetry is clearly too restricted. Therefore the author considers in this article not only symmetry preserving facetings, but sub-symmetrical ones as well. It will be shown that this produces a vast plenitude of interesting new polyhedra. Unusual symmetries as chiral tetrahedral (see left) do occur just as polyhedra which open their circumcenter to the access from outside, even getting toroids (see last figure). On the other hand, in order to keep that immense task manageable, the author had to restrict the kind of accepted sub-symmetries somewhat a priori. They are chosen to show up at least some rotational axis of order greater than 2. However, the computational steps given later on, would apply to the full spectrum alike.

Regiments and Colonels

For getting order in a plenitude of things, names derived from military seem quite appropriate. George Olshevsky defined regiments of any n-dimensional polytope (and thus especially of any polyhedron for n=3) to be the set of all those n-dimensional polytopes which have the same edge skeleton. For instance the icosahedron (ike) and the great dodecahedron (gike) do belong to the same regiment.

The leading representative of a regiment he calls its colonel. It is defined to be that polyhedron which is as convex as possible, i.e. having convex vertex figures only. So it is either the convex hull, if that is a member of the regiment, or at least some locally convex polyhedron. (For the latter class is just defined by that requirement.)

From the definition all members of a regiment have the symmetry of the colonel. The author applies those terms in a slightly weaker form in order to get those classes to contain sub-symmetrical members too, asking only that the skeleton of the members is at least a subset of that of their colonel.

From these concepts we get the following systematic for the set of uniform polyhedra:

Colonel	Edges per vertex	Further uniform members of its regiment	Colonel	Edges per vertex	Further uniform members of its regiment	Colonel	Edges per vertex	Further uniform members of its regiment
tet	3	-	ti	3	-	gid	4	gidhid, geihid
tut	3	-	tid	3	-	tiggy	3	-
oct	4	thah	srid	4	saddid, sird	gosid	5	-
cube	3	-	grid	3	-	quit sissid	3	-
co	4	oho, cho	snid	5	-	quitdid	3	-
toe	3	-	sidtid	6	ditdid, gidtid	isdid	5	-
tic	3	-	siid	4	sidditdid, siddy	gaddid	4	qrid, gird
sirco	4	socco, sroh	seside	6	-	quit gissid	3	-
girco	3	-	sissid	5	gike, starp *)	gaquatid	3	-
snic	5	-	did	4	sidhei, gidhei	gisid	5	-
gocco	4	querco, groh	tigid	3	-	sirsid	6	-
cotco	3	-	raded	4	ided, ri	girsid	5	-
quith	3	-	siddid	5	-	gidrid	6	gisdid *)
quitco	3	-	gidditdid	4	giid, giddy	pip	3	-
ike	5	gad, pap *)	idtid	3	-	stip	3	-
doe	3	-	sided	6	-	stap	4	-
id	4	seihid, sidhid	gissid	3	-			

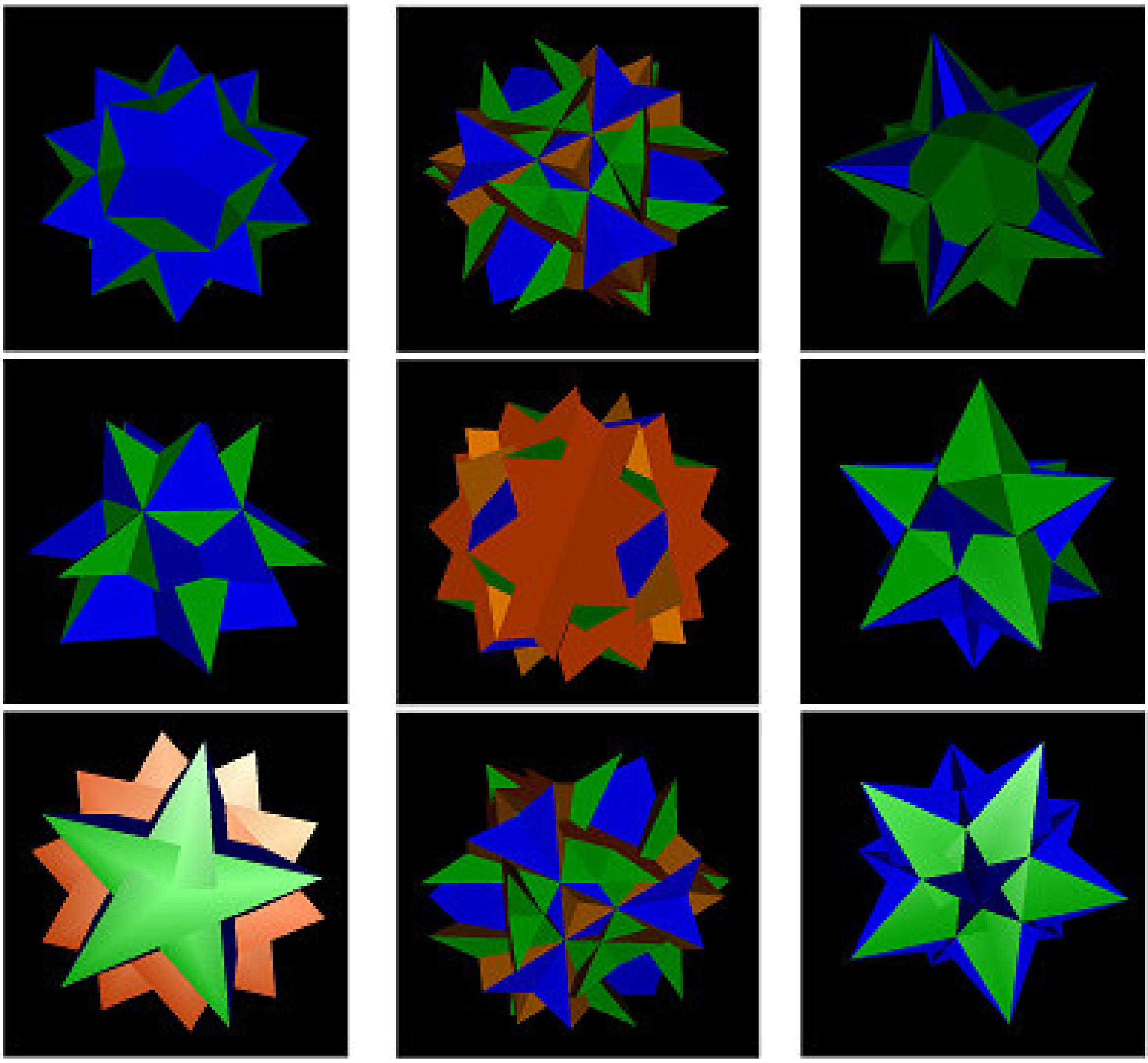
*) Those thus marked do show already some sort of sub-symmetry.

Algorithm

First we note that the research only has to be applied to regiments instead of all uniform polyhedra. Further, for those with 3 edges per vertex it is senseless to be applied.

<u>Step 1</u>: Apply numbers to all the vertices of the colonel.

<u>Step 2</u>: Describe all edges of the colonel by pairs of vertex numbers.



gid-6-10-1 gaddid-6-6-6-12 sissid-6-10

<u>Step 3</u>: Describe all faces of the colonel and all other planar faces which could be described from the edges of step 2 by circuits of vertex numbers.

Step 4: Choose a symmetry (either the full symmetry of the colonel, or any sub-symmetry thereof). Although in principle it is possible to use any one, it suffices to consider the minimal ones only, if the facetings found are later on reconsidered with respect to their actual symmetry. In the set-up of this article it reduces to the chiral pyramidal ones.

Step 5: Apply new labels to classes of symmetry-equivalent vertices.

Step 6: Apply new labels to classes of symmetry-equivalent edges.

Step 7: Apply new labels to classes of symmetry-equivalent faces (of any kind) from step 3.

Step 8: Specify extra symmetry-operations which would transform to be found facetings into ones which describe (essentially) the same polyhedron (for instance central inversion for axial sub-symmetries; as the faceting would be considered the same whether it will be positioned top up or top down; or extra mirrors for chiral sub-symmetries which would identify enantiomorph pairs).

Step 9: For each extra symmetry of step 8 set up a bijection of labels for the face-classes of step 7 which would be transformed into one another.

Step 10: Having enumerated the face-classes (step 7, call their total count f) use now bits for representation of each single class, and use binary numbers as representation for any set of those classes: i.e. 2^(1-1)=1=00001, 2^(2-1)=2=00010, 2^(3-1)=4=00100 etc. describes the first, second, third etc. of say f=5 possible classes alone, and further 9=01001 would thus describe the union of classes numbered 1 and 4. This will leave us with a range of (2^f)-1 binary numbers for all combinations of face-classes. The -1 is applied as the binary number 0=00000, which would denote no face-class being selected at all, and which clearly can be omitted.

Step 11: Set up a class relation between the edge-classes of step 6 and the face-classes of step 7: For each member of a specific edge-class there is some number of incident faces, which belong to some face-classes: either 0 or 1 or 2. To each other member of that same edge-class belong some other incident faces, but the corresponding numbers are still the same as before.

Step 12: For each binary number from the appropriate range of step 10 evaluate for each edge-class the count of those face-classes which lie in the intersection of the set of face-classes attributed to that given edge-class by the relation of step 11 on the one hand, and of the set of face-classes described by that binary number on the other hand. Therefore the evaluation count of that algorithm would amount to c=((2^f)-1)*e incidence evaluations, were e is the count of edge classes and f the count of face classes.

Remark: Note that the range f is the most crucial one with respect for the count c. The highest it gets is for the regiment of the great dirhombicosidodecahedron i.e. Miller's monster. Then additional investigations such as a splitting into well-chosen sub-cases had to be applied in order for the task to remain manageable.

Step 13: Mark any binary number of the range of step 10 as valid, iff all evaluations of step 12 yield either "0" (no face incident to that edge) or "2" (exactly 2 faces incident to that edge).

Step 14: List all valid numbers together with their reverse interpretation i.e. face classes.

Step 15: Reduce that list from all entries which are either to be identified by the bijections of step 9, or are unions from other entries (i.e. compounds of more elementary facetings), or were already given within the corresponding list for a higher symmetry.

Remark: The remaining entries are all the different (in sense of step 8) edge-facetings of the colonel which have a specific symmetry (chosen at step 4).

Addendum: The specific set-up of these steps (esp. step 3) makes it very easy to get for any derived edge-faceting the set of used faces, each in terms of vertex-numbers. Those numbers are requested by the program Hedron in the input files to produce VRML graphics of those facetings. Best results are produced, if all unused faces are listed in the corresponding input files as well, just marked to be "blind" faces. (Those blind faces are used by the software while iterating the polyhedron to its final form, but are not be given in the final face list in the VRML-export.)

The nomenclature of edge-facetings used in this article and the webpage is the name of the colonel followed by the counts of faces in the order given in the next table. If required it is followed by some version extension in the order they were derived.

Exemplified Application

Often it is much more enlighting to see some algorithm in action than just giving it mere abstractly. So it will be applied here to the small stellated dodecahedron (sissid) in the 5-fold symmetry.

Step 1:The convex hull of sissid is the icosahedron (ike). So it has 12 vertices. Here they will be numbered hexadecimal: 1 at the top, 2 through 6 running around the upper hemisphere, and the other ones being the complements to 13 for each opposite pair of vertices.

Step 2: In this arrangement the 30 edges of sissid, which are the extended edges of the dodecahedral kernel, can be described as the following hexadecimal pairs (vertices given in ascending order): 17, 18, 19, 1a, 1b, 24, 25, 27, 2a, 2c, 35, 36, 39, 3b, 3c, 46, 48, 4a, 4c, 57, 59, 5c, 68, 6b, 6c, 79, 7a, 8a, 8b, 9b.

It should be mentioned nevertheless that this numbering with hexadecimal base was chosen in the article just for notational reasons, but Hedron is not able to accept characters for numerical reasons, as it uses already characters to give additional advises to the program.

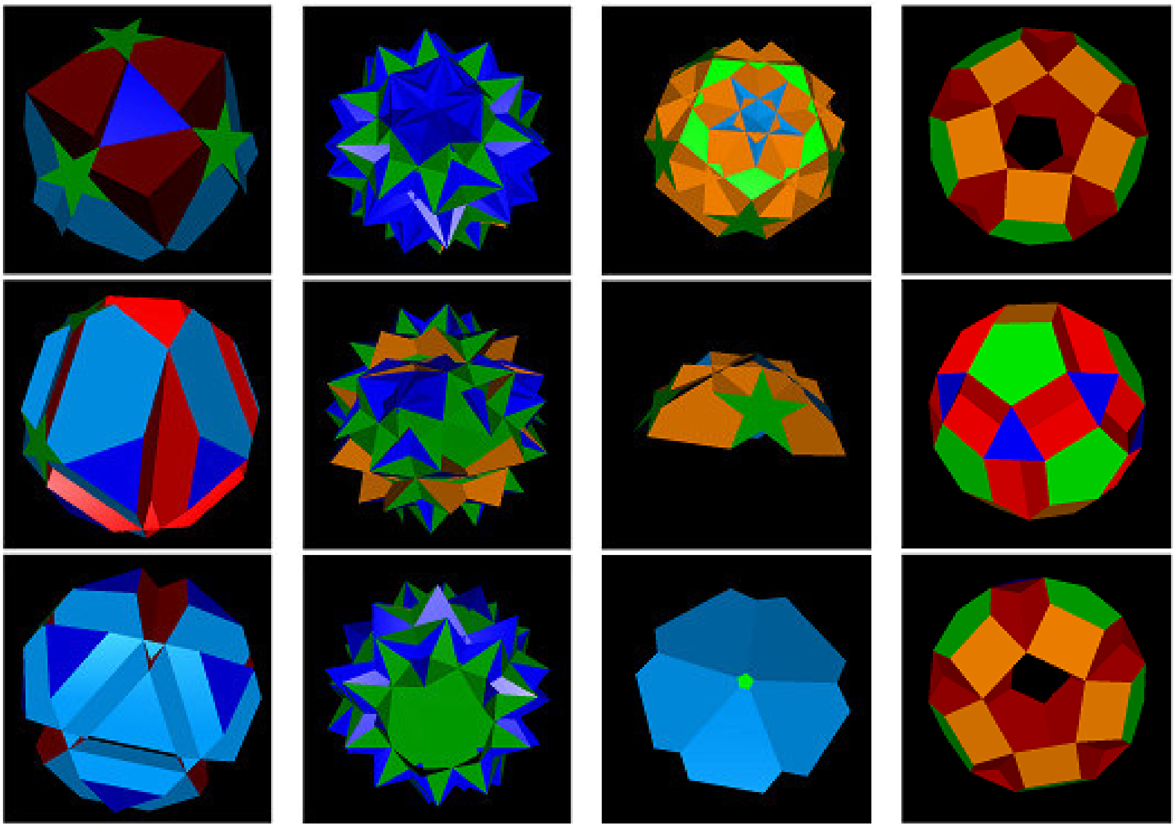
Step 3: The faces of sissid are 12 pentagrams. They will be given by hexadecimal pentuples in the sequence as the vertices are aligned by edges. Thus those circuits are cut open and oriented such that the smallest possible hexadecimal number appears. They are: 17248, 1753b, 18639, 1952a, 1a46b, 24635, 2793c, 2a86c, 3b84c, 4a75c, 59b6c, 79b8a.

The vertex figure of the sissid is a regular pentagon. Its vertices can be joined alternatingly as well, resulting in a pentagram, scaled with respect to the former by the golden ratio number tau = 2 cos(36°) = (1+√5)/2 ≈ 1.618034. Those larger lines of the vertex figure correspond to triangular faceting faces, connecting for instance the top vertex with the lower hemispherical pentagram. There are 20 such triangles, which, taken alone, make up the great icosahedron (gike). Denotation alike the former faces give: 179, 17a, 18a, 18b, 19b, 24a, 24c, 257, 25c, 27a, 359, 35c, 36b, 36c, 39b, 468, 46c, 48a, 579, 68b.

Step 4: As mentioned above the 5-fold (chiral) pyramidal sub-symmetry is chosen.

Step 5: Under the symmetry chosen in step 4 vertices fall obviously into 4 classes. Names for edge classes follow the systematic: alphabetical enumerator for the class, number for the order of rotational symmetry, “1” for vertex. We have:

A51 = {1},
B51 = {2, 3, 4, 5, 6},
C51 = {7, 8, 9, a, b},
D51 = {c}.



siid-6-10-10-6-d gidrid-24-60-10-v1 raded-5-10-1-5 srid-10-10-10-10

Step 6: Under the symmetry chosen in step 4 edges fall into 6 classes: those incident to vertex 1, those of the upper hemispherical pentagram, those connecting upper and lower hemispherical vertices excluding the polar ones (these falling into 2 enantiomorph classes), those of the lower pentagram, and those incident to c. Names for edge classes follow the same systematic as in step 5, with a final “2” for edge. Enantiomorph classes are given the same character, getting distinguished instead by a final prime. Thus we have:

A52 = <A51, C51> = {17, 18, 19, 1a, 1b},
B52 = <B51, B51> = {24, 25, 35, 36, 46},
C52 = <B51, C51>|type1 = {27, 3b, 4a, 59, 68},
C52’ = <B51, C51>|type2 = {2a, 39, 48, 57, 6b},

D52 = <C51, C51> = {79, 7a, 8a, 8b, 9b},
E52 = <B51, D51> = {2c, 3c, 4c, 5c, 6c}.

Step 7: Applying the same principle to the faces with final “3” for triangles respective “5” for pentagrams we get:
A53 = <A51, 2x C51> = <2x A52, D52> = {179, 17a, 18a, 18b, 19b},
B53 = <2x B51, C51> = <B52, C52, C52’> = {24a, 257, 359, 36b, 468},
C53 = <B51, 2x C51> = <C52, C52’, D52> = {27a, 39b, 48a, 579, 68b},
D53 = <2x B51, D51> = <B52, 2x E52> = {24c, 26c, 35c, 36c, 46c}
A55 = <5x B51> = <5x B52> = {24635}
B55 = <A51, 2x B51, 2x C51> = <2x A52, B52, C52, C52’>
= {17248, 1753b, 18639, 1952a, 1a46b}
C55 = <2x B51, 2x C51, D51> = <C52, C52’, D52, 2x E52>
= {2793c, 2a86c, 3b84c, 4a75c, 59b6c}
D55 = <5x C51> = <5x D52> = {79b8a}

Steps 8 & 9: As given in the algorithm this is the central inversion, interchanging A53 and D53, B53 and C53, A55 and D55, B55 and C55. Additional mirrors (containing the axis of symmetry) have not to be considered, as although enantiomorph edges do exist, the face classes are all automorph with respect to this kind of symmetry.

Step 10: In step 7 we got 8 classes of faces, i.e. f = 8. So we will need 8 bits for representation. Use A53 = 10000000 = 128, B53 = 01000000 = 64, C53 = 00100000 = 32, D53 = 00010000 = 16, A55 = 00001000 = 8, B55 = 00000100 = 4, C55 = 00000010 = 2, D55 = 00000001 = 1. Any dual number of 8 bits thus represents a 5-fold symmetric collection of corresponding faces.

Step 11: Face classes, incident to the edge classes, are:
A52: 2x A53, 2x B55
B52: B53, D53, A55, B55
C52: B53, C53, B55, C55
C52’: B53, C53, B55, C55
D52: A53, C53, C55, D55
E52: 2x D53, 2x C55

We see that e = 6. But as faces disallow chirality classes C52 and C52’ could be unified, and thereby reduce this count.

Step 12 & 13: Corresponding to the relations of step 11 we have 6 (or rather 5) functions defined for the set of numbers 1 = 00000001 till 255 = 11111111:
f_A52(x) = 2x bit 1 + 2x bit 6

f_B52(x) = bit 2 + bit 4 + bit 5 + bit 6
f_C52(x) = bit 2 + bit 3 + bit 6 + bit 7 = f_C52'(x)
f_D52(x) = bit 1 + bit 3 + bit 7 + bit 8
f_E52(x) = 2x bit 4 + 2x bit 7

Numbers are considered to be valid iff each of those functions evaluate either to 0 or 2.

Step 14+15: Valid are the 14 numbers:

00001111	= 15	= A55 + B55 + C55 + D55	= sissid-12-0	= sissid
00011000	= 24	= D53 + A55	(inverse to 129)	
00100010	= 34	= C53 + C55	(inverse to 68)	
00101101	= 45	= C53 + A55 + B55 + D55	(inverse to 75)	
00110101	= 53	= C53 + D53 + B55 + D55	(inverse to 202)	
01000100	= 68	= B53 + B55	= sissid-5-5	
01001011	= 75	= B53 + A55 + C55 + D55	= sissid-7-5	
01101001	= 105	= B53 + C53 + A55 + D55	= sissid-2-10	= 5/3-antiprism
01110001	= 113	= B53 + C53 + D53 + D55	(inverse to 232)	
10000001	= 129	= A53 + D55	= sissid-1-5	= 5/2-pyramid
10011001	= 153	= A53 + D53 + A55 + D55	(compound of 24 and 129)	
11001010	= 202	= A53 + B53 + A55 + C55	= sissid-6-10	
11101000	= 232	= A53 + B53 + C53 + A55	= sissid-1-15	
11110000	= 240	= A53 + B53 + C53 + D53	= sissid-0-20	= gike

Numbers 15 and 240 are icosahedral, number 105 has antiprismal symmetry, the remaining 5 are of (full) pyramidal symmetry.

The names in the fourth column just as those in the figures spread over the article are concatenations of the name of the colonel plus the counts of faces of different shapes aligned with increasing edge angle. If needed an additional isomer count is given at the end.

Statistics

From the given table of regiments those entries with 3 edges per vertex are trivial. There will be no further edge-faceting than the polyhedron itself. The other empty entries are snubs. A short analysis in the sense of step 3 applied to the snubs shows that only the gisdid allows additional faces (being itself a chiral edge-faceting of gidrid). For all other snubs the regiment is again trivial, even in its sub-symmetrical sense. The remaining 16 regiments are considered below explicitly. Only non-compound ones are listed in the last column.

Colonel	Possible faces	Symmetry of the facetings	e *)	f *)	c *)	Count of facetings **)
oct	8 triangles, 3 squares	octahedral 4-fold - pyramidal 3-fold - tetrahedral	1 3 4	1+1 2+2 4+1	3 45 124	1 (oct) 1 (4pyr) 1 (thah)
co	8 triangles, 6 squares, 4 hexagons	octahedral 4-fold 3-fold - pyramidal	1 6 8	1+1+1 2+3+1 4+2+2	7 378 2,040	3 (co, oho, cho) 0 2 (for inst.: 3cup)
sirco	8 triangles, 18 squares, 6 octagons	octahedral 4-fold - prismatic - pyramidal 3-fold - pyramidal	2 12 16	1+2+1 2+6+3 4+6+2	30 24,564 65,520	3 (sirco, socco, sroh) 3 (for inst.: 8p=op) 4 (for inst.: 4cup=J4, J19) 2
gocco	6 octagrams, 8 triangles, 6 squares	octahedral 4-fold - prismatic - pyramidal 3-fold - pyramidal	2 12 16	1+1+2 3+2+6 2+4+6	30 24,564 65,520	3 (gocco, querco, groh) 3 (for inst.: 8/3-p = stop) 4 (for inst.: 4/3-cup) 2
ike	20 triangles, 12 pentagons	icosahedral 5-fold - antiprismatic - pyramidal 3-fold - antiprismatic - pyramidal	1 6 10	1+1 4+4 8+4	3 1,530 40,950	2 (ike, gad) 1 (5ap = pap) 5 (for inst.: 5pyr=J2, J11) 2 3 (for inst.: J63)
id	20 triangles, 12 pentagons, 6 decagons	icosahedral 5-fold - pyramidal 3-fold - pyramidal	1 12 20	1+1+1 4+4+2 8+4+2	7 12,276 327,660	3 (id, sidhid, seihid) 2 (for inst.: J6) 2
srid	20 triangles, 30 squares, 12 pentagons, 12 decagons	icosahedral 5-fold - antiprismatic - pyramidal 3-fold - antiprismatic - antiprismatic	2 24 40	1+1+1+1 4+6+4+4 8+10+4+4	30 6,291,432 2,684,354,52 0	3 (srid, saddid, sird) 3 (for inst.: J80) 10 (for inst.: 5cup=J5, J76) 3 11 (for inst.: J83)
sidtid	12 pentagrams, 20 triangles, 30 squares, 12 pentagons	icosahedral 5-fold - antiprismatic - pyramidal 3-fold - octahedral - antiprismatic - chiral antiprismatic - chiral tetrahedral - pyramidal - chiral pyramidal	1 12 20	1+1+1+1 4+4+6+4 4+8+10+4	15 3,145,716 1,342,177,26 0	3 (sidtid, ditdid, gidtid) 7 46 (for inst.: 5/4-, 5/2-cupid) 1 (cube) 6 6 1 ***) 72 12
siid	12 pentagrams, 20 triangles, 20 hexagons, 12 decagons	icosahedral 5-fold - pyramidal 3-fold - pyramidal	2 24 40	1+1+1+1 4+4+4+4 4+8+8+4	30 1,572,840 671,088,600	3 (siid, sidditdid, siddy) 6 6
sissid	12 pentagrams, 20 triangles	icosahedral 5-fold - antiprismatic	1 6	1+1 4+4	3 1,530	2 (sissid, gike) 1 (5/3-ap=starp)

Colonel	Possible faces	Symmetry of the facetings	e *)	f *)	c *)	Count of facetings **)
		- pyramidal				5 (for inst.: 5/2-pyr)
		3-fold	10	4+8	40,950	
		- antiprismatic				2
		- pyramidal				3
did	12 pentagrams,	Icosahedral	1	1+1+1	7	3 (did, sidhei, gidhei)
	12 pentagons,	5-fold	12	4+4+2	12,276	
	10 hexagons	- pyramidal				2
		3-fold	20	4+4+4	81,900	
		- pyramidal				2
raded	12 pentagrams,	icosahedral	2	1+1+1+1	30	3 (raded, ided, ri)
	30 squares,	5-fold	24	4+6+4+4	6,291,432	
	12 pentagons,	- antiprismatic				2
	20 hexagons	- pyramidal				12
		3-fold	40	4+10+4+8	2,684,354,52	
		- antiprismatic			0	4
		- pyramidal				12
gidditdid	20 triangles,	icosahedral	2	1+1+1+1	30	3 (gidditdid, giid, giddy)
	12 decagrams,	5-fold	24	4+4+4+4	1,572,840	
	12 pentagons,	- pyramidal,				6
	20 hexagons	3-fold	40	8+4+4+8	671,088,600	
		- pyramidal				6
gid	12 pentagrams,	Icosahedral	1	1+1+1	7	3 (gid, gidhid, geihid)
	20 triangles,	5-fold	12	4+4+2	12,276	
	6 decagrams	- pyramidal				2
		3-fold	20	4+8+2	327,660	
		- pyramidal				2
gaddid	12 pentagrams,	icosahedral	2	1+1+1+1	30	3 (gaddid, qrid, gird)
	20 triangles,	5-fold	24	4+4+4+6	6,291,432	
	12 decagrams,	- antiprismatic				3
	30 squares	- pyramidal				10 (for inst.: 5/3-cup)
		3-fold	40	4+8+4+10	2,684,354,52	
		- antiprismatic			0	3
		- pyramidal				11
gidrid	24 pentagrams,	icosahedral	4	2+4+1	508	
	160 triangles,	- full icosahedral				1 (gidrid)
	60 squares	- chiral				
		icosahedral				1 (gisdid)
		5-fold	48	8+32+12	2.16173 *e+17*	
		- antiprismatic				15
		- chiral				
		antiprismatic				9
		- rotation-reflectiv				
		****)				120
		- pyramidal				10
		- chiral pyramidal				250
		3-fold	80	8+56+20	1.54743 *e+27*	
		- with 24 {5/2}				117,798 *****)
		- without {5/2}				2 (oct, thah)

*) see step 12 for those numbers; e and f being the counts of symmetry-different edge resp. face classes, c being the total evaluation count. Numbers e and f are chosen as for the least, i.e. the chiral pyramidal symmetry; all facetings with higher symmetry will thereby detected too.

**) Only non-compound facetings are counted. Chiral pairs are counted just once. Numbers are given as NEI: not elsewhere included (a term coined by N. Johnson). The suffixes –pyr, -cup, -cupid stand for pyramid, cupola, and cuploid. J## are the numbers of the Johnson solids.

***) This one is displayed above as sidtid-0-8-12-0-b.

****) This is an other rather rarely mentioned symmetry. Alike the chiral antiprismatic one it is a chiral axial symmetry. They are easily opposed by an investigation from the side:

… a – b – c – a – b – c …		… a – b – c – a – b – c …	
… – c – a – b – c – a – …	rotation-reflection	… c – b – a – c – b – a …	chiral antiprismatic

*****) That huge number is neither further analysed nor pictures are provided. For all other edge-facetings VRMLs are given at the webside, and for all but the 2 other larger groups of that colonel, JPEGs are shown too.

Count management

A short look on the huge count numbers c of the last table at gidrid (also called: Millers monster) shows the extent of this research. If you could set up a computer aided research which would run approximately 1 million combinations of step 10 per second, in the 5-fold case you thus would have to run your computer 140 years of pure calculation time! This clearly would be impossible to wait for in one human life span. For the 3-fold case the problem is even worse, it would amount to 6 x 10^{11} years, neglecting the slightly increased count of edge classes. Using further the time scale back to the big bang (Hubble time), the pure calculation time would thus be 47 times as large … - So it should be obvious that some additional information has to be used in order to get that monster done.

The first possible reduction amounts in a factor of 2^7 for both the 5-fold and the 3-fold cases. These can be achieved by the investigation of the vertex figure. That figure shows that either both possible incident pentagrams have to be chosen simultaneously at a vertex or none. Now, using this information and forwarding it from vertex (-class) to vertex (-class), one comes to see that all pentagrams have to lie in just a single face (meta-) class.

Next reductions can be achieved by splitting into cases with or without pentagrams. If no pentagrams are used, the remaining face classes split right into 4 disconnected sets in the 5-fold case, respectively 8 disconnected sets in the 3-fold set. As compounds are to be excluded from the research it will be enough to apply the algorithm to those much smaller sets separately instead. The actual f-counts are 4 times 0+8+3 (5-fold) respectively 6 times 0+8+3 plus 2 times 0+4+1.

For the other part, i.e. those edge-facetings of gidrid which incorporate pentagrams, the splitting is again the same as before, only that those split groups of classes are reconnected via the pentagrams. Thus in this case one does not have to look for separate closed solutions in each separated set; one will have to look instead for such solutions in each part which leave the pentagrammic edges unclosed. The final solutions will then be obtained from the set of combinations of one partial solution each, which are added to the 24 pentagrams. And again, as said in the algorithm, that set of combinations has to be reduced from symmetry equivalencies or remaining compounds finally.

NOTES

J. Bowers acronyms: see http://members.aol.com/hedrondude/polyhedra12.html (and followings)

Colonel, Regiment: see http://members.aol.com/Polycell/glossary.html

Edge-facetings, providing the complete set of pictures as JPEG (3 views each) and VRML (interactive): see http://www.polyhedrix.de/e_klintro.htm , produced by the author, hosted by U. Mikloweit. (The labelled pictures of this article are inverse gray-scalings from rather few of those JPEGs.)

Hedron software by J. McNeill, used to produce the VRMLs (the JPEGs being produced as screenshots thereof): see his article this publication or at http://web.ukonline.co.uk/polyhedra/hedron.html

"Uniform polyhedra": article by H. S. M. Coxeter, M. Longuet-Higgins, and J. Miller, in: Philosophical Transactions of the Royal Society of London, Ser. A 246: pp.401-450., (1954).