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SNUBS, ALTERNATED FACETINGS, & STOTT-COXETER-DYNKIN DIAGRAMS

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(2001) *Convex Segmentochora*. In: Symmetry: Cult. & Sci., vol. 11, nos. 1-4, pp. 139-181.

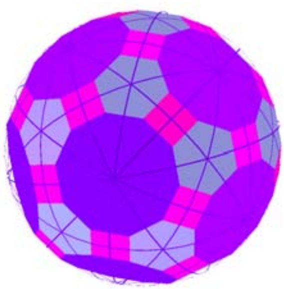
(1996) *Reskalierungssymmetrien quasiperiodischer Strukturen*. [*Symmetries of Rescaling for Quasiperiodic Structures*, Ph.D. Dissertation, in German], Eberhard-Karls-Universität Tübingen, or Hamburg, Verlag Dr. Kovač, ISBN 978-3-86064-428- 7. – (Former ones: mentioned therein.)

Abstract: *The snub cube and the snub dodecahedron are well-known polyhedra since the days of Kepler. Since then, the term "snub" was applied to further cases, both in 3D and beyond, yielding an exceptional species of polytopes: those do not bow to Wythoff's kaleidoscopical construction like most other Archimedean polytopes, some appear in enantiomorphic pairs with no own mirror symmetry, etc. However, they remained stepchildren, since they permit no real one-step access. Actually, snub polytopes are meant to be derived secondarily from Wythoffians, either from omnitruncated or from truncated polytopes. – This secondary process herein is analysed carefully and extended widely: nothing bars a progress from vertex alternation to an alternation of an arbitrary class of sub-dimensional elements, for instance edges, faces, etc. of any type. Furthermore, this extension can be coded in Stott-Coxeter-Dynkin diagrams as well.*

Keywords: polytopes, snubs, hemiations/semiations, alternated facetings, Johnson holosnubs, Wythoff kaleidoscopical construction, Stott-Coxeter-Dynkin diagrams, Stott expansion, any mixture of node symbols.

1. HISTORICAL REVIEW

This article assumes Wythoff's kaleidoscopical construction [*Wythoff, 1918*] to be known. As a brief recall that amounts to a set of N mirrors in N-dimensional space, all incident to a single point (which then becomes the centre of symmetry), enclosing a spherical simplex. Next a relative position of a seed-point is chosen. Each mirror will reflect that one into a mirror image. The point and its images will become (some of the) vertices of the to be constructed polytope, the line segments connecting the seed point to either image point become (some of) its edges, etc. Finally the complete flag, incident to the seed point, will be reflected into a flag centred at the mirror image point, and so the total polytope is constructed by iterated reflections whenever the symmetry group, generated by that mirror arrangement, is finite. Note that different such kaleidoscopes (together with their own seed-point) might produce the same polytope, for instance the cuboctahedron occurs both within the tetrahedral and the octahedral symmetry group. Polytopes, which are constructible by some kaleidoscope, further on will be called Wythoffian.



•	un-ringed nodes represent mirrors with vertices being on. (Transcription to inline ASCII-art: "o".)
⦿	ringed nodes represent mirrors with vertices being offshore. (In inline ASCII-art: "x".)
○	empty rings originally represented "missing" mirrors, again with vertices being offshore (cf. text for extensions). Those are Mrs. **Stott's snub nodes**. (In inline ASCII-art: "s", shortening both, 'snub' and 'Stott'.)
k	link, numbered k, represents a dihedral mirror angle of size π/k. Some cases often are further simplified: links marked 2 are completely omitted, links marked 3 usually lack that number. (In inline ASCII-art drawn links are completely represented by their numbers.)

Figure 1 Exemplified Wythoff's kaleidoscopical construction of the great rhombicosidodecahedron. Great circles represent the mirrors. Look at any spherical triangle and its seed-point (the vertex), which here is off with respect to all mirrors. That triangle has angles of sizes $\pi/2$, $\pi/3$ and $\pi/5$. Thus its Stott-Coxeter-Dynkin diagram is "x3x5x". & **Listing 1**: Graphical elements of the Stott-Coxeter-Dynkin diagrams. – Notations referred to as "inline ASCII-art" will be used in textual contexts as typewriter-friendly inline transcriptions.

Any Wythoffian polytope furthermore can be described by at least one graphical advice, which nowadays most commonly is called Coxeter-Dynkin diagram [*CD diagram*]. Here it will be assumed to be familiar with that one too. However, a short reference of its graphical elements is given in Listing 1. A detailed introduction Coxeter provides in his book "Regular Polytopes" [*Coxeter, 1948*]: There he starts with the symbols of Schläfli. Those, until then only useful for regular polytopes, become extended to reflection groups with non-linear graphs and to rectified polytopes of any order (a.k.a. quasi-regulars). In historical regard those extended Schläfli symbols just anticipated an appropriately bent group graph, transcribed into the syntax of Schläfli symbols. Further he provides a truncation-operator notation, which acts on these symbols. With that he has a fully valid precursor of the Coxeter-Dynkin diagram introduced later. Only thereafter he introduces Dynkin [*Dynkin*] graphs, modified slightly while transferring from Lie groups to polytopal reflection groups: mirrors still are represented by nodes, but links are not differentiated any longer by different link-symbols, they bear numbers as marks instead. These numbers equate to the divisor of π of the relevant dihedral angle between the two related mirrors. In addition, these symmetry group diagrams become decorated by rings around some nodes in a one-to-one correspondence to the above truncation-operator. This reveals that any Wythoffian polytope can be described by Coxeter-Dynkin diagrams. In fact, Coxeter-Dynkin diagrams, as far as described, bijectively encode the kaleidoscopical constructions, and therefore always give rise to a Wythoffian polytope.

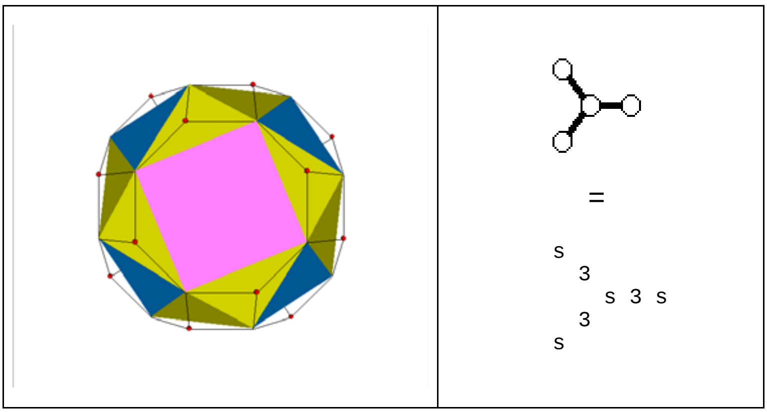


Figure 2a "s3s4s" as alternation of "x3x4x" – for the re-scaling of edges cf. text. (Then it would become the snub cube.) & **Figure 2b** The Stott-Coxeter-Dynkin symbol for the snub of the demi-tesseractic group.

1.1 Snubs

Snubs, known as such as a direct translation of Kepler's terms "cubus simus" and "dodekaedrum simum" [*Kepler, 1619*], so far could not be described by this mirror-based symbolism. In view of the enlisting of the set of uniform polyhedra by Coxeter, Longuet-Higgins, and Miller [*Coxeter et al., 1954*] (i.e. having just a single symmetry equivalent vertex type), it is apparent that this was a great lack in the symbolic notation of these days. It was Boole-Stott who added to this notation, allowing to include snubs also in that diagrammal transcription [*Boole-Stott, 1910*]: As snubs often lack mirror symmetry, the relevant new symbol uses empty rings when the vertices still are off the former mirror planes, but the nodes (i.e. mirrors) themselves are no longer suitable. Therefore the thus extended notation should rather be called Stott-Coxeter-Dynkin diagrams. (In this article this prefix henceforward will be abbreviated to SCD.)

1.1.1 Semiation of omnitruncates

Inline ASCII-art [*ASCII-art*], see Listing 1, represents the snub cube by "s3s4s" and the snub dodecahedron by "s3s5s". Recall that "s3s4s" has half as many vertices as "x3x4x" (omnitruncated cube), likewise "s3s5s" has half as many as "x3x5x" (omnitruncated dodecahedron). So, in fact, those are kind of semiations. Semi, being Latin for half; hemi would be its Greek counterpart. Coxeter clearly prefered hemiation, and as an operator (applied to his extended Schläfli symbols) he uses h. The author here prefers semi, as it more nicely relates to the s node, to snub, and to Stott.

1.1.2 Other semiations

Later Coxeter also comes up [*Coxeter, 1940/1985/1988*] with the symbol "s3s4o" for the semiation of "x3x4o" (truncated octahedron) or "s4o3o" as the semiation of "x4o3o" (cube). What are these? In fact those are just meant as notational shortcuts using the equivalence of represented solids, viz. "x3x4o" equates to "x3x3x". The latter one semiates to "s3s3s" (icosahedron). Accordingly, the symbolic semiation of the former, "s3s4o", is meant to be the same. And "x4o3o" equates to "x2x2x" (in here the 2's are superfluous, but for snubs those 2's imply, that semiation applies on the whole figure – opposed to an orthogonal product of semiated parts). So "s4o3o" equates to "s2s2s" (tetrahedron).

Coxeter [*Coxeter, 1948*] rediscovered Gosset's [*Gosset, 1900*] snub 24-cell. So "s3s4o3o", "s3s3s4o", and that in Figure 2b depicted non-linear graph all describe that same semiation of the truncated 24-cell (which in turn is represented by the same symbols, if "s" would be replaced by "x") with cells being either truncated octahedra or cubes: All these semiations, taken as alternation device, work alike when at least one of those can be shown to work. He finally notes [*Coxeter, 1940/1985/1988*] that snubbing in the sense of hemiation of truncates is much more general, even so in higher dimensions the count of different snub-edges, being resized to the same length, exceeds the available degree of freedom: But any symbol with up to 2 snub-nodes at some end, while the rest being empty nodes, guarantees equal sized edges, so does work (provided that the link between "s" and "o" bears an even number). Further, the sub-graph of the un-ringed nodes must have simplicial symmetry, if the snub is additionally asked to be uniform. – The 1st demand indeed relates to the degrees of freedom; the 2nd, in parentheses, is because else the application of the alternation around this special polygon does not close.

1.2 Holosnubs

Right here Johnson steps beyond Coxeter in introducing holosnubs [*Johnson, 1966*]. In fact, on a localized level those work exactly as normal snubs, alternating maintained vertices with ones being replaced by their vertex figure. But their SCD symbol does include at least one such offending odd digit, i.e. an odd-numbered polygonal circuit. To close the circuit none-the-less correctly, he just asks it to run twice around, resulting in doing both, maintaining and replacing each vertex. – This is, for a normal snub, a 2n-gon by mere local vertex alternation becomes a vertex inscribed n-gon. Holosnubs can alternate $(2n + 1)$-gons too, a pentagon, for instance, thus becomes a vertex inscribed pentagram. – Thus the vertex count, being halved for normal snubs, for holosnubs remains unchanged.

Alternatively one could think of holosnubs kind of like normal snubs, but replacing afore the to be alternated starting figure by a Grünbaumian double-cover [*Grünbaum, 2003*] of the former: a pentagon first would become a decagon $(10 = 5 \times 2)$ with winding number 2, so the alternation results then (quite normally) in one of 2 possible (coincident) pentagons $(5 = 10 : 2)$, still with winding number 2, i.e. the required pentagram.

Including holosnubs into consideration, the 2nd restriction of Coxeter's rule becomes obsolete, snub- and "o" nodes need no longer be separated by even marked links only. – Just to *visually* distinguish their different behaviours (e.g. concerning the vertex count etc.), the author uses “s” for normal snubs and “ß” for holosnubs as a mere additional service for the reader. The distinction else would nonetheless be derivable from the SCD symbol alone: depending on that existence of an odd link mark between snub and non-snub nodes.

1.2.1 Further symbolical extensions

There can be found SCD diagrams using simultaneously both, snub nodes and ringed nodes, e.g. on Olshevsky’s convex uniform polychora web page [Olshevsky]. This usage also roots back to Johnson [*Johnson, 1966*]. Just as in the given cases of Coxeter snubs (using un-ringed nodes), this defacto is still no true generalization of so far SCD diagrams. It rather is kind of a further notational ease, which applies exclusively to the cases “s4o3…”. Listing 2 shows the used definitions.

```
x                     x                      o
 3                     3                      3
  x 3 … = o4x3x3…       o 3 … = o4x3o3…        x 3 … = o4o3x3…
 3                     3                      3
x                     x                      o

o                     x                      x
 3                     3                      3
  o 3 … = o4o3o3…       x 3 … = “s4o3x3…”      o 3 … = “s4o3o3…”
 3                     3                      3
o                     o                      o
```

Listing 2 Johnson’s extension – purely for notational ease.

But exactly these symbols of Listing 2, re-read in the right way, give rise for a true generalization of Stott’s rules toward any mixture of snub and non-snub nodes, independent of being ringed or not. This will be the topic of this article, and especially of the next chapter. In the 3rd chapter then the implications thereof will be discussed.

2. EXTENSION OF NOTATION

There are two ways to understand the so far described (holo-)snubbing process as such. Either one starts with some polytope with a non-snub SCD diagram, does first some individual variation of edge lengths (which in its amount surely is implied only a posteriori) and applies thereafter the alternated faceting, leading directly to the uniform (holo-)snub, or at least one with equal-sized edges. Alternatively one reverses that order, applying alternated faceting first, which will be possible in any case, and looks for a variation of the result towards equal-sized edges thereafter. While Coxeter and Johnson insist on the former view (sometimes moreover including the truncation operation from a regular figure to the relevant Wythoffian starting figure), the author prefers the second one. More arguments to that point will be given later.

For the mere alternation neither a restriction to the form of SCD diagram is to be made, nor to the position of the snub nodes: The linkage of the graph can be whatever is allowed for non-snubs too, and for node symbols *any mixture of all three*, i.e. of ringed nodes (⦿ / "x"), un-ringed nodes (• / "o"), and snub-nodes (○ / "s" or "ß") is allowed. Thereby the usage of "s" (normal snub) or "ß" (holosnub) only depends on the graph, i.e. on the distribution of link mark numbers with respect to snub and non-snub node positions. Note: this extension to any mixture is completely new, and (except of the mere symbolic application in the former chapter) no occurrence of such generalized figures is known so far. – So, what does such an arbitrary mixture of SCD diagram constituents mean? Let's consider in the following for a first example the case "s4o3x" (cf. Figure 3a). That one also conforms to the already mentioned symbolic application (cf. Listing 2) and thus its outcome will be comparable to that older reading. But please keep in mind: in contrast to that older reading, which applies in few individual occasions only, the below provided completely independent interpretation would equally well apply to any such mixture of constituents.

2.1 Instructions

2.1.1 Derivation of Starting Figure

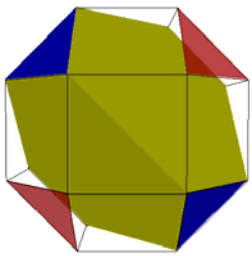
The first step derives from a given symbol (i.e. the extended arbitrary SCD diagram) that of the corresponding starting figure, done by replacing all snub-nodes by ringed nodes instead. For our chosen example ("s4o3x") this would result in "x4o3x", describing the small rhombicuboctahedron, which still is displayed in Figure 3a by narrow lines.

2.1.2 Derivation of Elements to be Alternated

Next we deduce what the elements to be alternated would be. Those are given by that sub-graph, where all snub-nodes (and their incident links) would be omitted. In our example ("s4o3x") this will be ". o3x", the triangles.

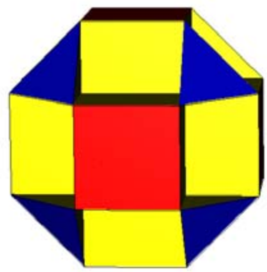
Just compare this situation to the snubs of the chapter “Historical Review” above. As introduced by Mrs. Stott, all nodes were empty rings. Therefore those will result, according to our instruction, in empty graphs for the elements to be alternated, i.e. in fact the vertices. In Coxeter’s extension of snub symbols, using un-ringed nodes in addition, that sub-graph alike would result in zero dimensional elements, i.e. vertices again. Accordingly, in these cases the replacement underneath the omitted element clearly is given by the vertex figure. In general, one would have to use quite similarly the sectioning facet right underneath the to be omitted element (which now does not need to be a vertex any longer). In our example this results in a semi-regular hexagonal facet underneath these triangles (as can be seen in Figure 3a).

In this case the resulting polyhedron can be made uniform, giving then the well-known truncated tetrahedron.



```
        s4o3x

demi( . . . ) | 12 | 1  2 | 1 2
--------------+----+------+----
      s4o .   |  2 | 6  * | 0 2
demi( . . x ) |  2 | * 12 | 1 1
--------------+----+------+----
demi( . o3x ) |  3 | 0  3 | 4 *
sefa( s4o3x ) |  6 | 3  3 | * 4
```



```
        ß3o4x

both( . . . ) | 24 |  2  2 | 1 1 2
--------------+----+-------+------
both( . . x ) |  2 | 24  * | 0 1 1
sefa( ß3o . ) |  2 |  * 24 | 1 0 1
--------------+----+-------+------
      ß3o .   |  3 |  0  3 | 8 * *
both( . o4x ) |  4 |  4  0 | * 6 *
sefa( ß3o4x ) |  8 |  4  4 | * * 6
```

Figure 3a “s4o3x” uses starting figure “x4o3x” and (snub-)alternates “. o3x”.
Figure 3b “ß3o4x” uses the same starting figure “x3o4x”, but holo-alternates “. o4x”.

2.1.3 Some Early Remarks

We use "sefa(…)" as operator to describe that required **se**ctioning **fa**cet underneath the alternated element. Thus even incidence matrices with explicit listings of relevant symbols can be given, cf. the lower half of the picture box of Figure 3a.

The same once more is visually outlined for the holosnub "ß3o4x", together with its incidence matrix, in Figure 3b. The remaining cases containing both "s"/"ß" and "x" nodes as decorations of the same cubical symmetry group diagram (that is to say the undecorated graph) are provided as pictures spread out in this article too. Pictures and/or incidence matrices of other snub polyhedra with a convex uniform starting figure can be obtained from the author on request.

Note that this extension not only covers Mrs. Stott's original aim and Coxeter's slightly extended usage, but also Johnson's purely notational application cited above. Thereby it not only reproduces the required figures (if edge resizement is added), but also provides intrinsic sense to these so far mere notational symbols. Even so the given rules are completely unique and straightforward (developed by the author in early 2005), they can be supported *a posteriori* in the view of Stott expansion [*Boole-Stott, 1910*] as well. For that purpose consider for instance again our example "s4o3x". In this view it can be separated into an initial Coxeter snub "s4o3o" and the expansion along ". . x", leading from the semiation of the cube ("x4o3o"), i.e. the tetrahedron, towards the truncated tetrahedron – just as required.

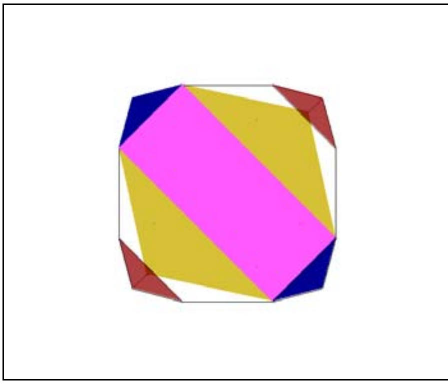
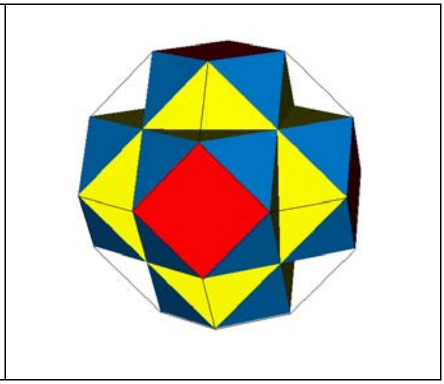



Figure 4a "s4x3o". & **Figure 4b** "ß3x4o".

The total count of Wythoffian decorations ("x" and "o" nodes) of a given undecorated

SCD graph with n nodes and no further intrinsic graph symmetry, is $2n - 1$ (the subtracted case with "o" nodes only would produce a non-relevant degenerate polytope: everything collapses into a single point, the center of symmetry, as in Wythoff's kaleidoscopical construction the seed-point has to be on all mirror at the same time). The provided extension of this chapter now adds $3n - 2n$ further possibilities, i.e. those with at least one snub node, to a total of $3n - 1$. All these added possibilities do exist as alternated facetings according to the constructive instruction part. Whether several ones do not have any resizement with equal-sized edges is an entirely different question.

3. DISCUSSION

Before going into the depth of discussions we will investigate some of the trickier alternated polygons in detail:

First, what is "ß3o"?

Let the vertices of a regular triangle be A, B, C. Vertices are to be alternated here. We start at A. So we maintain A, reject B, go ahead to C, which is maintained again, then, in a second circuit, now reject A, run ahead to B, which is maintained, further reject C, and run back to A, being maintained. Thus the loop closes. What is left over? A path from A to C, from C to B, and from B back to A, i.e. the same triangle in reverse order. – This can be seen also this way: Consider "ßNo", the alternated faceting of an odd N-gon. For larger N it is obvious that the result would be a vertex-inscribed "x N/2 o", so from a continuity argument the same should hold for "ß3o", yielding "x 3/2 o", i.e. that retrograde triangle. – Btw., "ß 5/2 o" by the same argument is "x 5/4 o", the retrograde convex pentagon, vertex-inscribed into the starting figure pentagram.

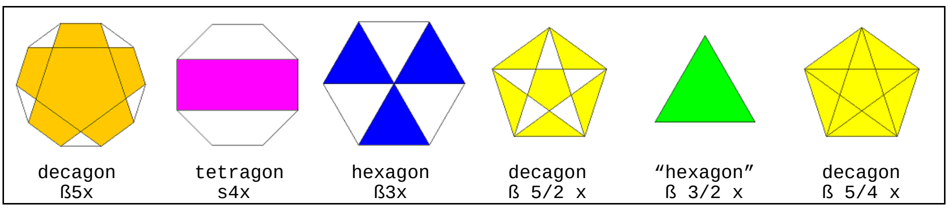


Figure 5 "sQx" & "ßQx". From left to right the "sefa"-edges run always deeper and do cross the center for Q = 3, thus thereafter looking retrograde with resp. to the maintained ". x"-edges. At Q = 3/2 those "sefa"-edges have zero length, i.e. turn over their orientation again.

So far only convex starting figures were discussed, but this alternated faceting likewise applies to non-convex polytopes. Whenever broken rationals occur for link marks, special care is needed:

Which polygon would be encoded by “ß 3/2 x”?

To answer this, instruction tells to look at its starting figure “x 3/2 x”. Just as any “xQx”, Q some rational number, taken as the described polygon, it will equate to “x 2Q o”, so too “x 3/2 x” equals to “x 6/2 o”, i.e. the regular hexagon with winding number 2. That one has six consecutive edges looping twice around the regular triangle. Its vertices are in positions A, B, C, A, B, C; but for clarity we rather use A1, B2, C3, A4, B5, C6. The edges all are prograde but here do alternate between 1st and 2nd type (“x .” respectively “. x”). Those of type 2, say, now will have to be alternated. Further it will be a holosnub, so we have to go twice through, which here means four times around that triangle. We will start at A1. A1B2 is an edge of type 1 and reduces to its end-point A1. B2C3, being of type 2, gets replaced by its “sefa”, in our case the “line” A1A4. C3A4 is of type 1, reduced to point A4. A4B5 is of type 2, this time being maintained. B5C6 is of type 1 and reduces to point B5. C6A1 is of type 2 and is replaced by the “line” B5B2. A1B2 is still of type 1, but in this circuit being reduced to its end-point B2. B2C3 is of type 2 and is maintained. C3A4 is of type 1 and gets reduced to point C3. A4B5 is of type 2 and gets replaced by the “line” C3C6. B5C6 is of type 1 and reduces to point C6. Finally closing, C6A1 is of type 2 and is maintained. Thus “ß 3/2 x” again has six “sides” AA, AB, BB, BC, CC, CA, but looks like a regular triangle, not even having some multiple winding. It here is just occasionally that every second side being of zero length.

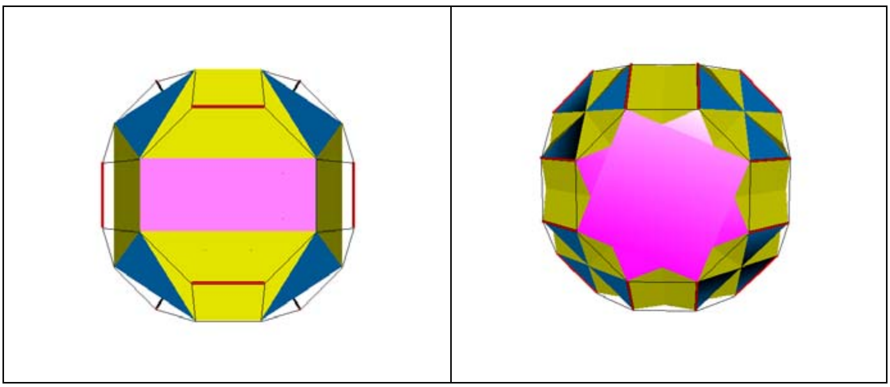


Figure 6a “s3s4x”. & **Figure 6b** “ß4ß3x”.

In fact, as can be seen from Figure 5, whenever "ßQx" has a rational $Q > 3$, the consecutive sides of the resulting polygon are all non-zero and both types, the maintained ones and the "sefa"-ones, are prograde. When $3 > Q > 3/2$, both sides are still non-zero, but the "sefas" have crossed the midpoint and so look retrograde with respect to the maintained ones. And at the just described instance of $Q = 3/2$ the "sefas" get zero length and so reverse thereafter their orientation again.

It follows, that for $Q \neq 3/2$ it still might be possible to produce a uniform variation of that resulting polygon. But for $Q = 3/2$ it would not be clear whether to use pro- or retrograde sides for those of zero length, or to reject them completely – what would create a non-continuity in Q. And what is even more problematic, that link mark 3/2 is found rather often in diagrams of non-elementary groups, i.e. those, which under kaleidoscopical construction give rise to non-convex starting figure polytopes. Therefore edge length resizement not only is not possible in several individual cases because of missing degrees of freedom, but it even *a priori* is an undefined operation in lots of symmetry groups!

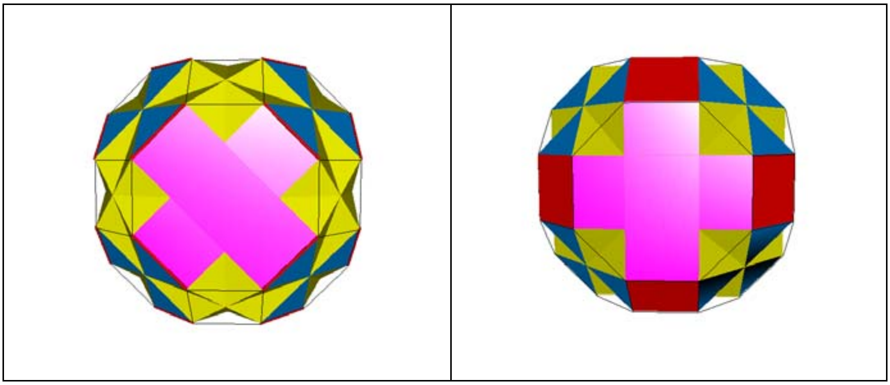


Figure 7a "ß3x4ß". & **Figure 7b** "x3ß4x".

For snubs and holosnubs with SCD diagrams without any ringed node it was mentioned that the vertex count will be halved (cf. "demi(…)") resp. remains the same (cf. "both(…)"). This behaviour is even true generally: Alternating any kind of sub-elements of a starting figure either simultanuously rejects or maintains all vertices incident to these elements. – For snubs, by the mere alternated faceting the symmetry of the figure is broken. For holosnubs (with no additional symmetry in its undecorated graph), because of simultanuously rejecting and maintaining the to be alternated elements, the symmetry will remain the same as for the starting figure, at least in the mere alternated faceting part. (If afterwards a variation will take place some additional symmetry might occure.)

4. CONCLUSION

Any snub, even according to the thus extended rules of SCD diagrams, now seems to be an alternated faceting. The main break-through of this article was won while bringing the historically required variation towards equal-sized edges (or even towards uniform snubs only) into the right order, i.e. onto the final step, and, for the matter of this article, being even suppressed. Both, the consequent usage of "mixed" symbols through all possible applications and an alternated faceting with respect to other elements than just vertices, were not known so far.

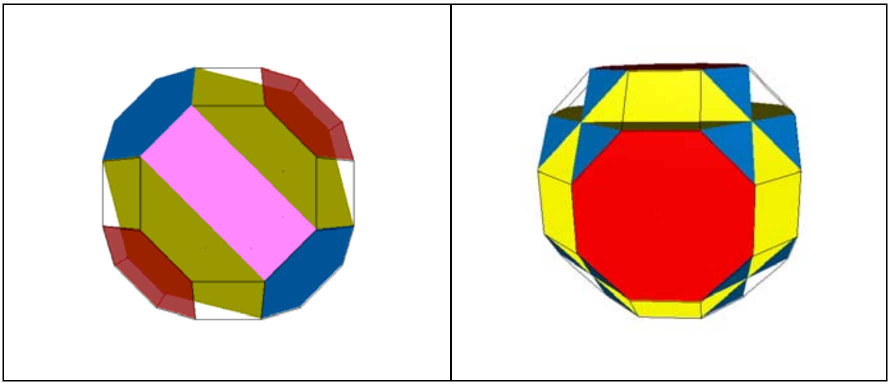


Figure 8a "s4x3x". & **Figure 8b** "ß3x4x".

More generally, faceting, being always considered as the dual operation of stellation (up to some exceptional singularities, conflicting with the usually understood definition of "polytope"), most often is found subordinate within the literature (esp. after the outstanding work "59 icosahedra" [*Coxeter et al., 1938*]). And even so, it only was applied with full symmetry of the starting figure. A single exception here is a previous article by the author [*Klitzing, 2002*]. Sub-symmetrical facetings usually are ignored. Thus "alternated faceting", except for vertex alternation (a.k.a. classical snubs), is a completely new idea.

What Mrs. Stott once described with her addition of empty rings to the SCD diagrams surely was intended to describe snubs in the sense of what later was called uniform polytopes, if only because of the choice of that “missing mirror” symbol. But in fact it much better matches the outlined alternated faceting of the corresponding Wythoffian starting figure. Whether a subsequent variation is possible or not should not be a question of the applicability of these snub nodes, but should be considered as a completely different kind of matter.

A symbolic distinction between normal snubs (“s”-nodes) and holosnubs (“ß”-nodes) is helpful for the recognition right on the first glance, but is already contained in the symbol elsewhere. So the non-ASCII type, i.e. graphical symbol (“○”) would be completely valid anyway.

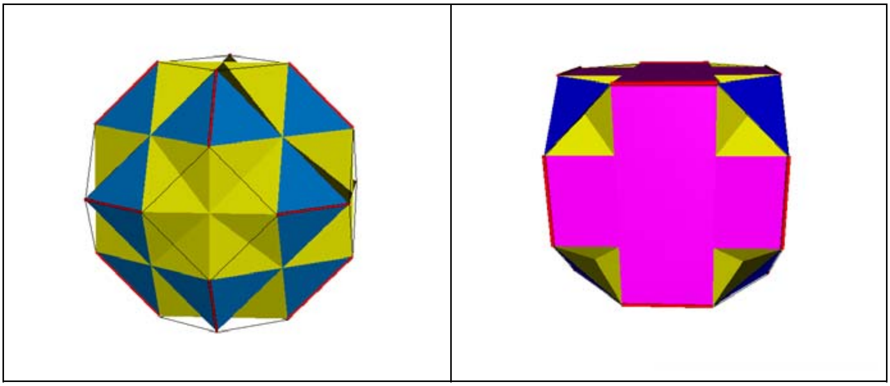


Figure 9a “x3ß4o”. & **Figure 9b** “x4ß3o”.

Finally it should be mentioned that there are two issues for this extended notation. On the one hand they describe a partially new class of polytopes, closely related to the snubs. (In low dimensions, the polytopes themselves not might be new, but their description. In higher ones to the contrary, the descriptional access only allows them to be considered at first.) On the second side occasionally there occurs some overlap to Wythoffians too. As an example the polyhedron of Figure 3b will be reconsidered here, the small cubicuboctahedron. That one not only can be recognized to be a uniform polyhedron, it also is Wythoffian. Its according SCD-diagram, which proves that statement, is provided in Listing 3 at the left. – In order to see the relations to the already provided other SCD-diagram of Figure 3b, the incidence matrix according to the now provided further symmetry group is given at the bottom of Listing 3. To that reason an additional inline ASCII-art convention was introduced by the author [*ASCII-art*], so that arbitrary diagrammal loops can be broken up to inline linearizations. He introduces besides the “real” nodes of the diagram (denoted by “x”, “o”, “s” or “ß”) additional "virtual" ones, occurring only within linearizations, which just refer to already elsewhere displayed spots. He counts all already displayed "real" nodes of a linearization from the left to the right alphabetically. Then, if having purpose to revisit such an already displayed node again, the linearization just will refer to that node (instead of doubling it) by its count-letter, prefixed by an (awareness-rising) asterix. Thus the diagram to the left of Listing 3 might be given linearized (i.e. broken up) as "x4x4o3/2*a", where "*a" refers to the left-most node. – But getting back to the main point to be said here: by the use of the extension to the snub notation it becomes possible to display some else rather awkwardly looped SCD-diagrams with rational link marks alternatively as a much nicer linear diagram with integral numbers only!

```
   x
      4
  3/2    x     =   ß 3 o 4 x
      4
   o


x4x4o3/2*a
. . .      | 24 |  2  2 | 2 1 1
-----------+----+-------+------
x . .      |  2 | 24  * | 1 1 0
. x .      |  2 |  * 24 | 1 0 1
-----------+----+-------+------
x4x .      |  8 |  4  4 | 6 * *
x . o3/2*a |  3 |  3  0 | * 8 *
. x4o      |  4 |  0  4 | * * 6
```

Listing 3 Two different SCD-diagrams of the small cubicuboctahedron. The left one shows that it truly is a Wythoffian polyhedron, the right re-displays the one provided in Figure 3b, showing that it can be given as a holosnub too. At the bottom the left SCD-diagram – broken up into a linearization (cf. text) – is used to provide the incidence matrix of the small cubicuboctahedron with respect to that symmetry group.

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[*ASCII-art*] *Inline ASCII-art for Stott-Coxeter-Dynkin diagrams*. Cf. for instance in W. Krieger's web page http://www.geocities.com/os2fan2/gloss/ptdynkin.htm [1], or on the authors web page http://ogre.nu/klitzing/explain/dynkin-notation.htm [2].

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[*CD diagram*] Variously being called *Dynkin graphs*, *Dynkin diagrams* [both only with respect to polytopes], *Coxeter-Dynkin diagrams*, or *Stott-Coxeter-Dynkin diagrams*. Cf. for example http://en.wikipedia.org/wiki/Coxeter-Dynkin_diagram .

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[1] *Meanwhile at http://os2fan2.com/gloss/ptdynkin*

[2] *Meanwhile at http://bendwavy.org/klitzing/explain/dynkin-notation.htm*